SECTION 1-4 Absolute Value in Equations and Inequalities

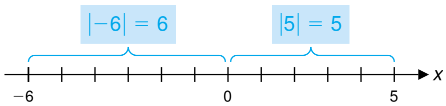
- Absolute Value and Distance
- Absolute Value in Equations and Inequalities
- Absolute Value and Radicals

This section discusses solving absolute value equations and inequalities.

• Absolute Value and Distance

We start with a geometric definition of absolute value. If a is the coordinate of a point on a real number line, then the distance from the origin to a is represented by $|a|$ and is referred to as the **absolute value** of a. Thus, $|5| = 5$, since the point with coordinate 5 is five units from the origin, and $|-6| = 6$, since the point with coordinate -6 is six units from the origin (Fig.1).

FIGURE 1 Absolute value.



Symbolically, and more formally, we define absolute value as follows:

DEFINITION 1

Absolute Value

$$|x| = \begin{cases} x & \text{if } x \geq 0 \quad |4| = 4 \\ -x & \text{if } x < 0 \quad |-3| \boxed{= -(-3)} = 3 \end{cases}$$

[*Note:* $-x$ is positive if x is negative.]

Both the geometric and nongeometric definitions of absolute value are useful, as will be seen in the material that follows. Remember:

The absolute value of a number is never negative.

EXAMPLE 1 **Absolute Value of a Real Number**

(A) $|\pi - 3| = \pi - 3$ Since $\pi \approx 3.14$, $\pi - 3$ is positive.
(B) $|3 - \pi| = -(3 - \pi) = \pi - 3$ Since $3 - \pi$ is negative

Matched Problem 1 Write without the absolute value sign:

(A) $|8|$ (B) $|\sqrt[3]{9} - 2|$ (C) $|-\sqrt{2}|$ (D) $|2 - \sqrt[3]{9}|$

Following the same reasoning used in Example 1, the next theorem can be proved (see Problem 79 in Exercise 1-4).

Theorem 1

For all real numbers a and b,

$$|b - a| = |a - b|$$

We use this result in defining the distance between two points on a real number line.

DEFINITION 2

Distance between Points *A* and *B*

Let A and B be two points on a real number line with coordinates a and b, respectively. The **distance between *A* and *B*** is given by

$$d(A, B) = |b - a|$$

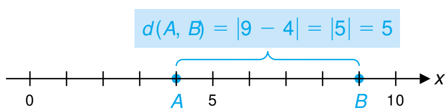
This distance is also called the **length of the line segment** joining A and B.

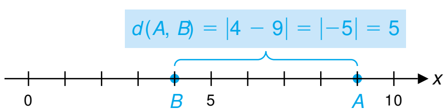
EXAMPLE 2 Distance between Points on a Number Line

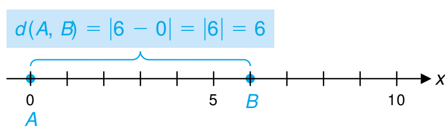
Find the distance between points A and B with coordinates a and b, respectively, as given.

(A) $a = 4$, $b = 9$ (B) $a = 9$, $b = 4$ (C) $a = 0$, $b = 6$

Solutions

(A) 

(B) 

(C) 

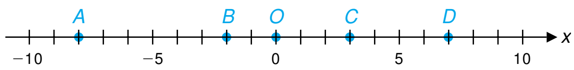
It should be clear, since $|b - a| = |a - b|$, that

$$d(A, B) = d(B, A)$$

Hence, in computing the distance between two points on a real number line, it does not matter how the two points are labeled—point A can be to the left or to the right of point B. Note also that if A is at the origin O, then

$$d(O, B) = |b - 0| = |b|$$

Matched Problem 2 Use the number line below to find the indicated distances.



(A) $d(C, D)$ (B) $d(D, C)$ (C) $d(A, B)$
(D) $d(A, C)$ (E) $d(O, A)$ (F) $d(D, A)$

• Absolute Value in Equations and Inequalities

The interplay between algebra and geometry is an important tool when working with equations and inequalities involving absolute value. For example, the algebraic statement

$$|x - 1| = 2$$

can be interpreted geometrically as stating that the distance from x to 1 is 2.

EXPLORE-DISCUSS 1 Write geometric interpretations of the following algebraic statements:

(A) $|x - 1| < 2$ (B) $0 < |x - 1| < 2$ (C) $|x - 1| > 2$

EXAMPLE 3 **Solving Absolute Value Problems Geometrically**

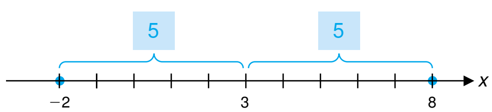
Interpret geometrically, solve, and graph. Write solutions in both inequality and interval notation, where appropriate.

(A) $|x - 3| = 5$ (B) $|x - 3| < 5$
(C) $0 < |x - 3| < 5$ (D) $|x - 3| > 5$

Solutions (A) Geometrically, $|x - 3|$ represents the distance between x and 3. Thus, in $|x - 3| = 5$, x is a number whose distance from 3 is 5. That is,

$$x = 3 \pm 5 = -2 \qquad \text{or} \qquad 8$$

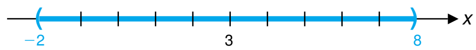
The solution set is $\{-2, 8\}$. This *is not* interval notation.



(B) Geometrically, in $|x - 3| < 5$, x is a number whose distance from 3 is less than 5; that is,

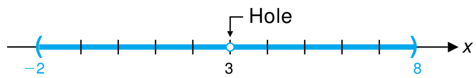
$$-2 < x < 8$$

The solution set is $(-2, 8)$. This *is* interval notation.



(C) The form $0 < |x - 3| < 5$ is frequently encountered in calculus and more advanced mathematics. Geometrically, x is a number whose distance from 3 is less than 5, but x cannot equal 3. Thus,

$$-2 < x < 8 \qquad x \neq 3 \qquad \text{or} \qquad (-2, 3) \cup (3, 8)$$



(D) Geometrically, in $|x - 3| > 5$, x is a number whose distance from 3 is greater than 5; that is,

$$x < -2 \qquad \text{or} \qquad x > 8 \qquad \text{or} \qquad (-\infty, -2) \cup (8, \infty)$$

CAUTION

Do not confuse solutions like

$$-2 < x \qquad \text{and} \qquad x < 8$$

which can also be written as

$$-2 < x < 8 \qquad \text{or} \qquad (-2, 8)$$

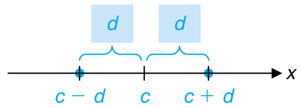
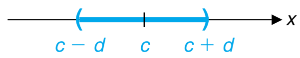
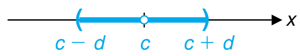
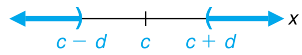
with solutions like

$$x < -2 \qquad \text{or} \qquad x > 8$$

which cannot be written as a double inequality or as a single interval.

We summarize the preceding results in Table 1.

TABLE 1 Geometric Interpretation of Absolute Value Equations and Inequalities

Form ($d > 0$)	Geometric interpretation	Solution	Graph
$\lvert x - c \rvert = d$	Distance between x and c is equal to d.	$\{c - d, c + d\}$	
$\lvert x - c \rvert < d$	Distance between x and c is less than d.	$(c - d, c + d)$	
$0 < \lvert x - c \rvert < d$	Distance between x and c is less than d, but $x \neq c$.	$(c - d, c) \cup (c, c + d)$	
$\lvert x - c \rvert > d$	Distance between x and c is greater than d.	$(-\infty, c - d) \cup (c + d, \infty)$	

Matched Problem 3 Interpret geometrically, solve, and graph. Write solutions in both inequality and interval notation, where appropriate.

(A) $\lvert x + 2 \rvert = 6$ (B) $\lvert x + 2 \rvert < 6$
(C) $0 < \lvert x + 2 \rvert < 6$ (D) $\lvert x + 2 \rvert > 6$

[*Hint:* $\lvert x + 2 \rvert = \lvert x - (-2) \rvert$.]

EXPLORE-DISCUSS 2

Describe the set of numbers that satisfies each of the following:

(A) $2 > x > 1$ (B) $2 > x < 1$
(C) $2 < x > 1$ (D) $2 < x < 1$

Explain why it is never necessary to use double inequalities with inequality symbols pointing in different directions. Standard mathematical notation requires that all inequality symbols in an expression must point in the same direction.

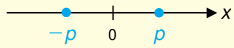
Reasoning geometrically as before (noting that $\lvert x \rvert = \lvert x - 0 \rvert$) leads to Theorem 2.

Theorem 2

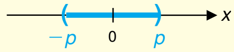
Properties of Equations and Inequalities Involving $|x|$

For $p > 0$:

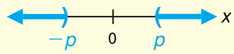
1. $|x| = p$ is equivalent to $x = p$ or $x = -p$.



2. $|x| < p$ is equivalent to $-p < x < p$.



3. $|x| > p$ is equivalent to $x < -p$ or $x > p$.



If we replace x in Theorem 2 with $ax + b$, we obtain the more general Theorem 3.

Theorem 3

Properties of Equations and Inequalities Involving $|ax + b|$

For $p > 0$:

1. $|ax + b| = p$ is equivalent to $ax + b = p$ or $ax + b = -p$.
2. $|ax + b| < p$ is equivalent to $-p < ax + b < p$.
3. $|ax + b| > p$ is equivalent to $ax + b < -p$ or $ax + b > p$.

EXAMPLE 4 Solving Absolute Value Problems

Solve, and write solutions in both inequality and interval notation, where appropriate.

(A) $|3x + 5| = 4$ (B) $|x| < 5$ (C) $|2x - 1| < 3$ (D) $|7 - 3x| \le 2$

Solutions

(A)
$$|3x + 5| = 4$$
$$3x + 5 = \pm 4$$
$$3x = -5 \pm 4$$
$$x = \frac{-5 \pm 4}{3}$$
$$x = -3, -\tfrac{1}{3}$$
or $\{-3, -\tfrac{1}{3}\}$

(B)
$$|x| < 5$$
$$-5 < x < 5$$
or $(-5, 5)$

(C) $|2x - 1| < 3$

$$-3 < 2x - 1 < 3$$
$$-2 < 2x < 4$$
$$-1 < x < 2$$

or $(-1, 2)$

(D) $|7 - 3x| \le 2$

$$-2 \le 7 - 3x \le 2$$
$$-9 \le -3x \le -5$$
$$3 \ge x \ge \tfrac{5}{3}$$
$$\tfrac{5}{3} \le x \le 3$$

or $[\frac{5}{3}, 3]$

Matched Problem 4 Solve, and write solutions in both inequality and interval notation, where appropriate.

(A) $|2x - 1| = 8$ (B) $|x| \le 7$ (C) $|3x + 3| \le 9$ (D) $|5 - 2x| < 9$

EXAMPLE 5 **Solving Absolute Value Inequalities**

Solve, and write solutions in both inequality and interval notation.

(A) $|x| > 3$ (B) $|2x - 1| \ge 3$ (C) $|7 - 3x| > 2$

Solutions

(A) $|x| > 3$

$x < -3$ or $x > 3$ Inequality notation

$(-\infty, -3) \cup (3, \infty)$ Interval notation

(B) $|2x - 1| \ge 3$

$2x - 1 \le -3$ or $2x - 1 \ge 3$

$2x \le -2$ or $2x \ge 4$

$x \le -1$ or $x \ge 2$ Inequality notation

$(-\infty, -1] \cup [2, \infty)$ Interval notation

(C) $|7 - 3x| > 2$

$7 - 3x < -2$ or $7 - 3x > 2$

$-3x < -9$ or $-3x > -5$

$x > 3$ or $x < \frac{5}{3}$ Inequality notation

$(-\infty, \frac{5}{3}) \cup (3, \infty)$ Interval notation

Matched Problem 5 Solve, and write solutions in both inequality and interval notation.

(A) $|x| \ge 5$ (B) $|4x - 3| > 5$ (C) $|6 - 5x| > 16$

EXAMPLE 6 **An Absolute Value Problem with Two Cases**

Solve: $|x + 4| = 3x - 8$

Solution Theorem 3 does not apply directly, since $3x - 8$ is not known to be positive. However, we can use the definition of absolute value and two cases: $x + 4 \geq 0$ and $x + 4 < 0$.

Case 1. $x + 4 \geq 0$ (that is, $x \geq -4$)
For this case, the possible values of x are in the set $\{x \mid x \geq -4\}$.

$$
\begin{aligned}
|x + 4| &= 3x - 8 \\
x + 4 &= 3x - 8 && |a| = a \text{ for } a \geq 0 \\
-2x &= -12 \\
x &= 6 && \text{A solution, since 6 is among the possible values of } x
\end{aligned}
$$

The check is left to the reader.

Case 2. $x + 4 < 0$ (that is, $x < -4$)
In this case, the possible values of x are in the set $\{x \mid x < -4\}$.

$$
\begin{aligned}
|x + 4| &= 3x - 8 \\
-(x + 4) &= 3x - 8 && |a| = -a \text{ for } a < 0 \\
-x - 4 &= 3x - 8 \\
-4x &= -4 \\
x &= 1 && \text{Not a solution, since 1 is not among the possible values of } x
\end{aligned}
$$

Combining both cases, we see that the only solution is $x = 6$.

Check As a final check, we substitute $x = 6$ and $x = 1$ in the original equation.

$$
\begin{aligned}
|x + 4| &= 3x - 8 & \qquad |x + 4| &= 3x - 8 \\
|6 + 4| &\stackrel{?}{=} 3(6) - 8 & |1 + 4| &\stackrel{?}{=} 3(1) - 8 \\
10 &\stackrel{\checkmark}{=} 10 & 5 &\neq -5
\end{aligned}
$$

Matched Problem 6 Solve: $|3x - 4| = x + 5$

• Absolute Value and Radicals

In Section A-7, we show that if x is positive or 0, then

$$\sqrt{x^2} = x$$

If x is negative, however, we must write

$$\sqrt{x^2} = -x \qquad \sqrt{(-2)^2} = -(-2) = 2$$

Thus, for x any real number,

$$\sqrt{x^2} = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

But this is exactly how we defined $|x|$ at the beginning of this section (see Definition 1). Thus, for x any real number,

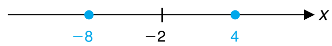
$$\sqrt{x^2} = |x|$$

Answers to Matched Problems

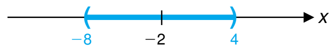
1. (A) 8 (B) $\sqrt[3]{9} - 2$ (C) $\sqrt{2}$ (D) $\sqrt[3]{9} - 2$

2. (A) 4 (B) 4 (C) 6 (D) 11 (E) 8 (F) 15

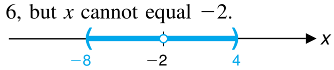
3. (A) x is a number whose distance from -2 is 6.
$x = -8, 4$ or $\{-8, 4\}$



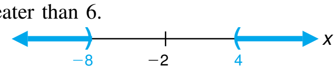
(B) x is a number whose distance from -2 is less than 6.
$-8 < x < 4$ or $(-8, 4)$



(C) x is a number whose distance from -2 is less than 6, but x cannot equal -2.
$-8 < x < 4$, $x \neq -2$, or $(-8, -2) \cup (-2, 4)$



(D) x is a number whose distance from -2 is greater than 6.
$x < -8$ or $x > 4$, or $(-\infty, -8) \cup (4, \infty)$



4. (A) $x = -\frac{7}{2}, \frac{9}{2}$ or $\{-\frac{7}{2}, \frac{9}{2}\}$ (B) $-7 \leq x \leq 7$ or $[-7, 7]$ (C) $-4 \leq x \leq 2$ or $[-4, 2]$
(D) $-2 < x < 7$ or $(-2, 7)$

5. (A) $x \leq -5$ or $x \geq 5$, or $(-\infty, -5] \cup [5, \infty)$ (B) $x < -\frac{1}{2}$ or $x > 2$, or $(-\infty, -\frac{1}{2}) \cup (2, \infty)$
(C) $x < -2$ or $x > \frac{22}{5}$, or $(-\infty, -2) \cup (\frac{22}{5}, \infty)$

6. $x = -\frac{1}{4}, \frac{9}{2}$ or $\{-\frac{1}{4}, \frac{9}{2}\}$

EXERCISE 1-4

A

In Problems 1–8, simplify, and write without absolute value signs. Do not replace radicals with decimal approximations.

1. $|\sqrt{5}|$

2. $|-\frac{3}{4}|$

3. $|(-6) - (-2)|$

4. $|(-2) - (-6)|$

5. $|5 - \sqrt{5}|$

6. $|\sqrt{7} - 2|$

7. $|\sqrt{5} - 5|$

8. $|2 - \sqrt{7}|$

In Problems 9–12, find the distance between points A and B with coordinates a and b respectively, as given.

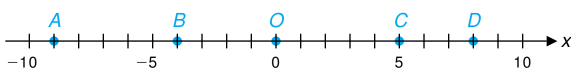
9. $a = -7$, $b = 5$

10. $a = 3$, $b = 12$

11. $a = 5$, $b = -7$

12. $a = -9$, $b = -17$

In Problems 13–18, use the number line below to find the indicated distances.



13. $d(B, O)$

14. $d(A, B)$

15. $d(O, B)$

16. $d(B, A)$

17. $d(B, C)$

18. $d(D, C)$

Write each of the statements in Problems 19–28 as an absolute value equation or inequality.

19. x is 4 units from 3.

20. y is 3 units from 1.

21. m is 5 units from -2.

22. n is 7 units from -5.

23. x is less than 5 units from 3.

24. z is less than 8 units from -2.

25. p is more than 6 units from -2.

26. c is no greater than 7 units from -3.

27. q is no less than 2 units from 1.

28. d is no more than 4 units from 5.

B

In Problems 29–44, solve, interpret geometrically, and graph. When applicable, write answers using both inequality notation and interval notation.

29. $|x| \le 7$ **30.** $|t| \le 5$ **31.** $|x| \ge 7$

32. $|x| \ge 5$ **33.** $|y - 5| = 3$ **34.** $|t - 3| = 4$

35. $|y - 5| < 3$ **36.** $|t - 3| < 4$ **37.** $|y - 5| > 3$

38. $|t - 3| > 4$ **39.** $|u + 8| = 3$ **40.** $|x + 1| = 5$

41. $|u + 8| \le 3$ **42.** $|x + 1| \le 5$ **43.** $|u + 8| \ge 3$

44. $|x + 1| \ge 5$

In Problems 45–62, solve each equation or inequality. When applicable, write answers using both inequality notation and interval notation.

45. $|3x - 7| \le 4$ **46.** $|5y + 2| \ge 8$

47. $|4 - 2t| > 6$ **48.** $|10 + 4s| < 6$

49. $|7m + 11| = 3$ **50.** $|4 - 5n| \le 8$

51. $|\frac{1}{2}w - \frac{3}{4}| < 2$ **52.** $|\frac{1}{3}z + \frac{5}{6}| = 1$

53. $|0.2u + 1.7| \ge 0.5$ **54.** $|0.5v - 2.5| > 1.6$

55. $|\frac{9}{5}C + 32| < 31$ **56.** $|\frac{5}{9}(F - 32)| < 40$

57. $\sqrt{x^2} < 2$ **58.** $\sqrt{m^2} > 3$

59. $\sqrt{(1 - 3t)^2} \le 2$ **60.** $\sqrt{(3 - 2x)^2} < 5$

61. $\sqrt{(2t - 3)^2} > 3$ **62.** $\sqrt{(3m + 5)^2} \ge 4$

C

Problems 63–66 are calculus-related. Solve and graph. Write each solution using interval notation.

63. $0 < |x - 3| < 0.1$ **64.** $0 < |x - 5| < 0.01$

65. $0 < |x - c| < d$ **66.** $0 < |x - 4| < d$

In Problems 67–76, for what values of x does each hold?

67. $|x - 2| = x - 2$ **68.** $|x + 4| = -(x + 4)$

69. $|2x - 3| = 3 - 2x$ **70.** $|3x - 9| = 3x - 9$

71. $|3x + 5| = 2x + 6$ **72.** $|7 - 2x| = 5 - x$

73. $|x| + |x + 3| = 3$ **74.** $|x| - |x - 5| = 5$

75. $|2x + 7| - |6 - 3x| = 8$ **76.** $|3x + 1| + |3 - 2x| = 11$

77. What are the possible values of $\dfrac{x}{|x|}$?

78. What are the possible values of $\dfrac{|x - 1|}{x - 1}$?

79. Prove that $|b - a| = |a - b|$ for all real numbers a and b.

80. Prove that $|x|^2 = x^2$ for all real numbers x.

81. Prove that the average of two numbers is between the two numbers; that is, if $m < n$, then

$$m < \frac{m + n}{2} < n$$

82. Prove that for $m < n$,

$$d\left(m, \frac{m + n}{2}\right) = d\left(\frac{m + n}{2}, n\right)$$

83. Prove that $|-m| = |m|$.

84. Prove that $|m| = |n|$ if and only if $m = n$ or $m = -n$.

85. Prove that for $n \ne 0$,

$$\left|\frac{m}{n}\right| = \frac{|m|}{|n|}$$

86. Prove that $|mn| = |m||n|$.

87. Prove that $-|m| \le m \le |m|$.

88. Prove the **triangle inequality:**

$$|m + n| \le |m| + |n|$$

Hint: Use Problem 87 to show that

$$-|m| - |n| \le m + \mathrm{n} \le |m| + |n|$$

89. If a and b are real numbers, prove that the maximum of a and b is given by

$$\max(a, b) = \tfrac{1}{2}[a + b + |a - b|]$$

90. Prove that the minimum of a and b is given by

$$\min(a, b) = \tfrac{1}{2}[a + b - |a - b|]$$

APPLICATIONS

91. Statistics. Inequalities of the form

$$\left|\frac{x - m}{s}\right| < n$$

occur frequently in statistics. If $m = 45.4$, $s = 3.2$, and $n = 1$, solve for x.

92. Statistics. Repeat Problem 91 for $m = 28.6$, $s = 6.5$, and $n = 2$.

★ **93. Business.** The daily production P in an automobile assembly plant is within 20 units of 500 units. Express the daily production as an absolute value inequality.

★ **94. Chemistry.** In a chemical process, the temperature T is to be kept within 10°C of 200°C. Express this restriction as an absolute value inequality.

95. Approximation. The area A of a region is approximately equal to 12.436. The error in this approximation is less than 0.001. Describe the possible values of this area both with an absolute value inequality and with interval notation.

96. Approximation. The volume V of a solid is approximately equal to 6.94. The error in this approximation is less than 0.02. Describe the possible values of this volume both with an absolute value inequality and with interval notation.

★ **97. Significant Digits.** If $N = 2.37$ represents a measurement, then we assume an accuracy of 2.37 ± 0.005. Express the accuracy assumption using an absolute value inequality.

★ **98. Significant Digits.** If $N = 3.65 \times 10^{-3}$ is a number from a measurement, then we assume an accuracy of $3.65 \times 10^{-3} \pm 5 \times 10^{-6}$. Express the accuracy assumption using an absolute value inequality.

SECTION 1-5 Complex Numbers

- Introductory Remarks
- The Complex Number System
- Complex Numbers and Radicals

• Introductory Remarks

The Pythagoreans (500–275 B.C.) found that the simple equation

$$x^2 = 2 \tag{1}$$

had no rational number solutions. If equation (1) were to have a solution, then a new kind of number had to be invented—an irrational number. The irrational numbers $\sqrt{2}$ and $-\sqrt{2}$ are both solutions to equation (1). Irrational numbers were not put on a firm mathematical foundation until the nineteenth century. The rational and irrational numbers together constitute the real number system.

Is there any need to consider another number system? Yes, if we want the simple equation

$$x^2 = -1$$

to have a solution. If x is any real number, then $x^2 \geq 0$. Thus, $x^2 = -1$ cannot have any real number solutions. Once again a new type of number must be invented, a number whose square can be negative. These new numbers are among the numbers called *complex numbers*. The complex numbers evolved over a long period of time, but, like the real numbers, it was not until the nineteenth century that they were placed on a firm mathematical foundation.

• The Complex Number System

We start the development of the complex number system by defining a complex number and several special types of complex numbers. We then define equality, addition, and multiplication in this system, and from these definitions the important special properties and operational rules for addition, subtraction, multiplication, and division will follow.