

# Teachers Teaching with Technology

T<sup>3</sup> Scotland



T<sup>3</sup> EUROPE

## Polynomials

# POLYNOMIALS

## *Aim*

To demonstrate how the TI-83 can be used to facilitate a fuller understanding of polynomials and show clearly the relationship between the algebraic solution and the graphical solution.

## *Objectives*

### **Mathematical objectives**

By the end of this unit you should know:

- how to recognise features of various polynomials
- the relationship between the graph of a situation and the algebra used to describe it
- the relationship between roots and factors
- how to factorise and solve polynomial equations
- how to find approximate roots of a polynomial by decimal search
- how to find polynomial coefficients

### **Calculator objectives**

By the end of this unit you should be able to

- draw graphs using [Y=]
- alter the display of a graph using [WINDOW] and [ZOOM].
- use the [2nd][TABLE] function with appropriate setting, using [2nd][TBLSET]

# Factorising and Solving Polynomial Equations

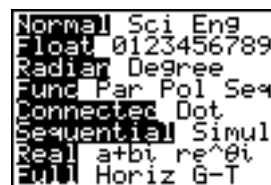
## Calculator skills sheet

Using a TI - 83 to assist you in solving polynomials not only reduces the chances of you making a silly error but also makes the whole process much faster. It can also help you gain a fuller understanding of the mathematics.

Here is a typical question from a textbook and a method for solution.

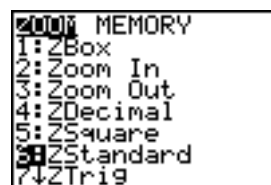
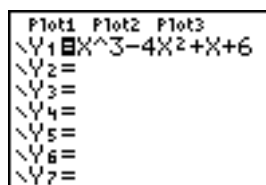
Find the roots of  $x^3 - 4x^2 + x + 6 = 0$

Before going any further check that the [MODE] screen looks like this, particularly ensure that the TI is not rounding answers by highlighting Float.



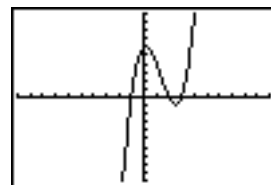
Rounded answers could appear as non-existent roots.

Enter the function on the [Y=] screen and graph the result on the [ZOOM 6:ZStandard] window range.



For some functions [ZOOM 4:ZDecimal] is appropriate.

The graph of this function shows that it has roots at values which look like -1, 2 and 3.



This can be confirmed by looking at a table of values over this range. Using the [2nd][TBLSET] screen and then [2nd][TABLE] we can see that the values of the function are indeed zero at -1, 2 and 3.



X	Y1
-2	-20
-1	0
0	6
1	4
2	0
3	0
4	10

X=4

Hence we can now say that  $x = -1$ ,  $x = 2$  and  $x = 3$  are roots.

Once we have obtained the roots from the calculator it is easy to say what the factors of the equation are and so this same method can be used to fully factorise a polynomial.

The process to be used is almost identical.

### Example 1

Find the roots of  $x^3 - 4x^2 + x + 6 = 0$

#### WORKED SOLUTION

From graphic calculator root at  $x = -1$

-1	1	-4	1	6	
		-1	5	-6	
	1	-5	6	0	← No remainder $x = -1$ is a root and $(x + 1)$ is a factor. (Factor Theorem)

↓

$x^2 - 5x + 6$  ← This quadratic must factorise to give  $(x - 2)$  and  $(x - 3)$ . We already know this from the graph and table but can confirm it by multiplying out the bracket or by factorising the quadratic

↓

$$\begin{aligned}x^3 - 4x^2 + x + 6 &= (x + 1)(x^2 - 5x + 6) = 0 \\ &= (x + 1)(x - 2)(x - 3) = 0 \\ \therefore x + 1 &= 0 \text{ or } x - 2 = 0 \text{ or } x - 3 = 0 \\ \therefore x &= -1 \text{ or } x = 2 \text{ or } x = 3\end{aligned}$$

### Exercise 1

Solve these Polynomial Equations.

1.  $x^3 - 6x^2 + 11x - 6 = 0$
2.  $x^3 + 4x^2 + 5x + 2 = 0$
3.  $x^3 - 3x^2 - 16x + 48 = 0$
4.  $x^4 - 3x^2 - 2x = 0$
5.  $x^4 - x^2 + 4x - 4 = 0$

### Example 2

Fully factorise  $x^3 - x^2 - x + 1$

This function is a cubic (polynomial of degree 3) which means it may have:

Roots

3 i.e.  $x = a, x = b, x = c$

2 i.e.  $x = a, x = b, x = b$

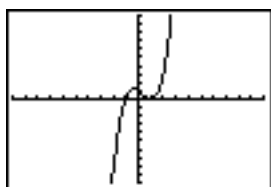
1 i.e.  $x = a$

Factors

3 Unique Linear Factors  $(x - a)(x - b)(x - c)$

1 Unique Linear Factor & 1 Repeated pair  $(x - a)(x - b)(x - b)$

1 Linear Factor & 1 Non-factorising Quadratic



[ZOOM 6:ZStandard]

X	Y1
-2	-9
-1	0
0	1
1	0
2	5
3	16
4	45

From the TI it can be seen that this function has 1 unique root and 1 pair of coincidental roots, i.e. it must have 1 unique linear factor and 1 pair of repeated linear factors. The table display shows that the roots are  $x = -1$  and  $x = 1$ , from the graph we can see that the roots at  $x = 1$ , are coincidental.

The roots of this function are  $x = -1, x = 1$  and  $x = 1$ .

The factors must be  $(x + 1)$  and  $(x - 1)$  and  $(x - 1)$

### WORKED SOLUTION

From graphic calculator root at  $x = -1$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & -1 & 1 \\ & & -1 & 2 & -1 \\ \hline & 1 & -2 & 1 & 0 \end{array}$$

$$\begin{array}{r} 1 \quad -2 \quad 1 \quad 0 \\ \downarrow \\ x^2 - 2x + 1 \end{array}$$

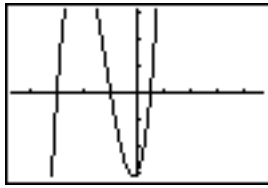
No remainder  $x = -1$  is a root and  $(x + 1)$  is a factor. (Factor Theorem)

This quadratic must factorise to give  $(x - 1)$  and  $(x - 1)$ .

$$\begin{aligned} x^3 - x^2 - x + 1 &= (x + 1)(x^2 - 2x + 1) \\ &= (x + 1)(x - 1)(x - 1) \end{aligned}$$

### Example 3

Fully factorise and solve  $2x^3 + 7x^2 + 2x - 3 = 0$



[ZOOM 4:ZDecimal]

X	Y1
-4	-27
-3	0
-2	5
-1	0
0	-3
1	8
2	45

X=-4

From the TI it can be seen that this function has 3 unique roots, i.e. it must have 3 linear factors. From the graph we can see that one of the roots is fractional (about  $x = 0.5$ ).

The table display confirms this showing roots at  $x = -3$ ,  $x = -1$  and another between  $x = 0$  and  $x = 1$ , since  $f(0) = -3$  and  $f(1) = 8$ .

TABLE SETUP	
TblStart=-1	
ΔTbl=.5	
Indent: Auto Ask	
Depend: Auto Ask	

X	Y1
-4	0
-3	-2.5
-2	-3
-1	0
0	8
1	22.5
2	45

X=-1

By changing the [2nd][TBL SET] as shown we can see that the fractional root is indeed at  $x = 0.5$

The roots of this function are  $x = -3$ ,  $x = -1$  and  $x = 0.5$

The factors must be  $(x + 3)$  and  $(x + 1)$  and  $(x - 0.5)$

#### WORKED SOLUTION

From graphic calculator root at  $x = -3$

$$\begin{array}{r|rrrr}
 -3 & 2 & 7 & 2 & -3 \\
 & & -6 & -3 & 3 \\
 \hline
 & 2 & 1 & -1 & 0
 \end{array}$$

No remainder  $\therefore x = -3$  is a root and  $(x + 3)$  is a factor. (Factor Theorem)

$$2x^2 + x - 1$$

This quadratic must factorise to give  $(x + 1)$  and  $(x - 0.5)$

We do not quote linear factors with fractional values so we can multiply the factor to eliminate the fractional value, in this case by 2 so  $(x - 0.5) = (2x - 1)$

Hence the factors of the resulting quadratic are:  $(x + 1)$  and  $(2x - 1)$

$$\begin{aligned}
 2x^3 + 7x^2 + 2x - 3 &= (x + 3)(2x^2 - x - 1) \\
 &= (x + 3)(x + 1)(2x - 1)
 \end{aligned}$$

$$\therefore x + 3 = 0 \quad \text{or} \quad x + 1 = 0 \quad \text{or} \quad 2x - 1 = 0$$

$$\therefore x = -3 \quad \text{or} \quad x = -1 \quad \text{or} \quad 2x = 1$$

$$\therefore x = -3 \quad \text{or} \quad x = -1 \quad \text{or} \quad x = \frac{1}{2}$$

## Approximate Roots of a Polynomial (Decimal Search)

### Example 1

Show that there is a real root of the equation  $x^3 + x - 3 = 0$

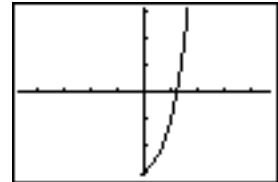
between 1 and 2, and find an approximation for the root correct to 2 decimal places.

On your TI enter the function and draw the graph.

[ZOOM 4:ZDecimal]

From this graph it can be seen that the function does have a root between  $x = 1$  and  $x = 2$

```
Plot1 Plot2 Plot3
\Y1=X^3+X-3
\Y2=
```



Using [2nd][TBL SET] set up a table of values as shown. The resulting table shows that the root does lie between  $x = 1$  and  $x = 2$

```
TABLE SETUP
TblStart=0
ΔTbl=1
Indent: Auto Ask
Depend: Auto Ask
```

X	Y1	Y2
0	-3	
1	-1	
2	7	
3	27	
4	65	
5	127	
6	219	

X=0

Using [2nd][TBL SET] change the starting value of the table and the step size.

The resulting table shows that the root does lie between  $x = 1.2$  and  $x = 1.3$

```
TABLE SETUP
TblStart=1
ΔTbl=.1
Indent: Auto Ask
Depend: Auto Ask
```

X	Y1	Y2
1	-1	
1.1	-.569	
1.2	-.072	
1.3	.497	
1.4	1.144	
1.5	1.875	
1.6	2.696	

X=1

Change the start and step size again.

The resulting table shows that the root does lie between  $x = 1.21$  and  $x = 1.22$

```
TABLE SETUP
TblStart=1.2
ΔTbl=.01
Indent: Auto Ask
Depend: Auto Ask
```

X	Y1	Y2
1.2	-.072	
1.21	-.0184	
1.22	.03585	
1.23	.09087	
1.24	.14662	
1.25	.20313	
1.26	.26038	

X=1.2

Change the start and step size once more.

The resulting table shows that the root does lie between  $x = 1.213$  and  $x = 1.214$ .

so to 2 d.p. the root is  $x = 1.21$

```
TABLE SETUP
TblStart=1.21
ΔTbl=.001
Indent: Auto Ask
Depend: Auto Ask
```

X	Y1	Y2
1.21	-.0184	
1.211	-.013	
1.212	-.0076	
1.213	-.0022	
1.214	.00319	
1.215	.00861	
1.216	.01405	

X=1.21

**WORKING**  $f(x) = x^3 + x - 3$

$$f(1) = -1$$

$$f(2) = 7$$

$$f(1.2) = -0.072$$

$$f(1.3) = 0.497$$

$$f(1.21) = -0.0184$$

$$f(1.22) = 0.03585$$

$$f(1.213) = -0.0022$$

$$f(1.214) = 0.00319$$

} Root between 1 and 2,  $1 < x < 2$

} Root between 1.2 and 1.3,  $1.2 < x < 1.3$

} Root between 1.21 and 1.22,  $1.21 < x < 1.22$

} Root is 1.2 to 1 d.p

} Root between 1.213 and 1.214,  $1.213 < x < 1.214$

} Root is 1.21 to 2 d.p

## Finding Polynomial Coefficients

### Example 1

If  $(x + 3)$  is a factor of  $2x^4 + 6x^3 + px^2 + 4x - 15$ , find the value of  $p$ .

On your TI enter the function using the letter  $P$  as the coefficient of  $x$  on the [Y=] screen.

Note to obtain the letter  $P$  use [ALPHA][P]

P1ot1	P1ot2	P1ot3
Y1=		
2X^4+6X^3+PX^2+4X-15		
Y2=		
Y3=		
Y4=		
Y5=		
Y6=		

The TI will evaluate the function for  $P=$  the value stored as  $P$  in the memory of the calculator.

Store the value zero as  $P$ . [0][STO][ALPHA][P][ENTER]

0→P	0
-----	---

Using [2nd][TBLSET] and [2nd][TABLE] obtain a value of this function when  $x = -3$ , the root which would accompany the factor  $(x + 3)$ .

The value of the function when  $P = 0$  and  $x = -3$  is  $-27$ .

Therefore the term  $Px^2$  must equal  $27$ ,

to make the value at  $x = -3$  be  $0$

If  $Px^2 = 27$ , when  $x = -3$  then  $P = 3$

X	Y1
-4	97
-3	-27
-2	-39
-1	-23
0	-15
1	-3
2	73

X=-4

This answer can now be checked by storing the value  $3$  as  $P$ .

[3][STO][ALPHA][P][ENTER]

0→P	0
3→P	3

The table of results now shows that  $x = -3$  is a root and therefore  $(x + 3)$  is a factor.

X	Y1
-4	145
-3	0
-2	-27
-1	-20
0	-15
1	0
2	85

X=-4

### WORKING

$$f(x) = 2x^4 + 6x^3 + Px^2 + 4x - 15$$

$$\text{Let } g(x) = 2x^4 + 6x^3 + 4x - 15 \quad \therefore f(x) = g(x) + Px^2$$

$$f(x) \text{ has a factor } (x + 3) \therefore f(x) \text{ has a root at } x = -3$$

$$\text{so} \quad f(-3) = g(-3) + P(-3)^2 = 0$$

$$\text{but } g(-3) = -27 \quad \therefore f(-3) = -27 + 9P = 0$$

$$9P = 27$$

$$P = 3$$



## Exercise 2

(Like Exercise 1 only trickier)

Solve these Polynomial Equations.

1.  $x^4 - 5x^2 + 4 = 0$
2.  $2x^4 - 9x^3 + 5x^2 - 3x - 4 = 0$
3.  $x^3 + x = 0$
4.  $x^3 - 19x + 30 = 0$
5.  $x^4 - 3x^2 - 2x = 0$
6.  $x^4 - 3x^2 + 2 = 0$

## Exercise 3

Fully factorise these polynomials.

1.  $x^3 - 7x + 6$
2.  $x^3 - 39x + 70$
3.  $x^3 - 8x^2 + 19x - 12$
4.  $2x^3 + 7x^2 + 2x - 3$
5.  $3x^3 - 4x^2 - 3x + 4$
6.  $3x^5 + 11x^4 + 5x^3 - 15x^2 - 8x + 4$

## Exercise 4

1. Show that  $x^3 + x^2 + 2x - 1 = 0$  has a root between 0 and 1.  
Find the value of this root to 1 d.p.
2. Show that  $x^4 + x^2 - 0.85 = 0$  has a root between 0 and 1, and another between -1 and 0.  
Find the values of these roots to 1 d.p.
3. Show that  $x^3 - x^2 - 2x + 1 = 0$  has a root between 1.5 and 2.  
Find the value of this root to 2 d.p.
4. Find all of the roots of  $9x^3 + 9x^2 - 9x - 7 = 0$  to 1 d.p.
5. Find all of the roots of  $9x^4 + 9x^2 - 9x - 7 = 0$  to 1 d.p.
6. Find all of the roots of  $9x^5 + 9x^2 - 9x - 7 = 0$  to 1 d.p.

## Exercise 5

1. Given that  $x + 3$  is a factor of  $x^3 - 3x^2 + kx + 6 = 0$   
Find the value of  $k$ .
2. Given that  $x + 1$  is a factor of  $x^4 + 4x^3 + ax^2 + 4x + 1 = 0$   
Find the value of  $a$ .
3. Given that  $x + 4$  is a factor of  $2x^4 + 9x^3 + 5x^2 + 3x + p = 0$   
Find the value of  $p$  and solve the equation.
4. Given that  $x^2 + 2x - 3$  is a factor of  $x^4 + 2x^3 - 7x^2 + px + q = 0$   
Find the value of  $p$  and  $q$ . Hence factorise this equation fully.  
(HINT: Try to form a pair of simultaneous equations)