

Lecture Notes in Mathematics

# An Introduction to Riemannian Geometry

Sigmundur Gudmundsson

(Lund University)

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## Preface

These lecture notes grew out of an M.Sc. course on differential geometry which I gave at the University of Leeds 1992. Their main purpose is to introduce the beautiful theory of Riemannian Geometry a still very active area of mathematical research. This is a subject with no lack of interesting examples. They are indeed the key to a good understanding of it and will therefore play a major role throughout this work. Of special interest are the classical Lie groups allowing concrete calculations of many of the abstract notions on the menu.

The study of Riemannian geometry is rather meaningless without some basic knowledge on Gaussian geometry that i.e. the geometry of curves and surfaces in 3-dimensional space. For this I recommend the excellent textbook: M. P. do Carmo, *Differential geometry of curves and surfaces*, Prentice Hall (1976).

These lecture notes are written for students with a good understanding of linear algebra, real analysis of several variables, the classical theory of ordinary differential equations and some topology. The most important results stated in the text are also proven there. Others are left to the reader as exercises, which follow at the end of each chapter. This format is aimed at students willing to put **hard work** into the course.

For further reading I recommend the interesting textbook: M. P. do Carmo, *Riemannian Geometry*, Birkhäuser (1992).

I am grateful to my many enthusiastic students who throughout the years have contributed to the text by finding numerous typing errors and giving many useful comments on the presentation.

Norra Nöbbelev, 17 February 2016

Sigmundur Gudmundsson



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## CHAPTER 1

### Introduction

On the 10th of June 1854 Georg Friedrich Bernhard Riemann (1826-1866) gave his famous "Habilitationsvortrag" in the Colloquium of the Philosophical Faculty at Göttingen. His talk "Über die Hypothesen, welche der Geometrie zu Grunde liegen" is often said to be the most important in the history of differential geometry. Johann Carl Friedrich Gauss (1777-1855), at the age of 76, was in the audience and is said to have been very impressed by his former student.

Riemann's revolutionary ideas generalized the geometry of surfaces which had been studied earlier by Gauss, Bolyai and Lobachevsky. Later this led to an exact definition of the modern concept of an abstract Riemannian manifold.

The development of the 20th century has turned Riemannian geometry into one of the most important parts of modern mathematics. For an excellent survey of this vast field we recommend the following work written by one of the main actors: Marcel Berger, *A Panoramic View of Riemannian Geometry*, Springer (2003).





## Differentiable Manifolds

In this chapter we introduce the important notion of a differentiable manifold. This generalizes curves and surfaces in  $\mathbb{R}^3$  studied in classical differential geometry. Our manifolds are modelled on the standard differentiable structure on the vector spaces  $\mathbb{R}^m$  via compatible local charts. We give many examples, study their submanifolds and differentiable maps between manifolds.

Let  $\mathbb{R}^m$  be the standard  $m$ -dimensional real vector space equipped with the topology induced by the Euclidean metric  $d$  on  $\mathbb{R}^m$  given by

$$d(x, y) = \sqrt{(x_1 - y_1)^2 + \dots + (x_m - y_m)^2}.$$

For a non-negative integer  $r$  and an open subset  $U$  of  $\mathbb{R}^m$  we shall by  $C^r(U, \mathbb{R}^n)$  denote the  $r$ -times continuously **differentiable** maps from  $U$  to  $\mathbb{R}^n$ . By **smooth** maps  $U \rightarrow \mathbb{R}^n$  we mean the elements of

$$C^\infty(U, \mathbb{R}^n) = \bigcap_{r=0}^{\infty} C^r(U, \mathbb{R}^n).$$

The set of **real analytic** maps from  $U$  to  $\mathbb{R}^n$  will be denoted by  $C^\omega(U, \mathbb{R}^n)$ . For the theory of real analytic maps we recommend the book: S. G. Krantz and H. R. Parks, *A Primer of Real Analytic Functions*, Birkhäuser (1992).

**Definition 2.1.** Let  $(M, \mathcal{T})$  be a topological Hausdorff space with a countable basis. Then  $M$  is called a **topological manifold** if there exists a positive integer  $m$ , for each point  $p \in M$  an open neighbourhood  $U$  of  $p$  and a continuous map  $x : U \rightarrow \mathbb{R}^m$  which is a homeomorphism onto its image  $x(U)$ . This is an open subset of  $\mathbb{R}^m$ . The pair  $(U, x)$  is called a (local) **chart** (or local **coordinates**) on  $M$ . The integer  $m$  is called the **dimension** of  $M$ . To denote that the dimension of  $M$  is  $m$  we write  $M^m$ .

According to Definition 2.1 a topological manifold  $M$  is locally homeomorphic to the standard  $\mathbb{R}^m$  for some natural number  $m$ . We shall now introduce a differentiable structure on  $M$  via its local charts and turn  $M$  into a differentiable manifold.

**Definition 2.2.** Let  $M$  be a topological manifold. Then a  $C^r$ -atlas on  $M$  is a collection

$$\mathcal{A} = \{(U_\alpha, x_\alpha) \mid \alpha \in I\}$$

of local charts on  $M$  such that  $\mathcal{A}$  covers the whole of  $M$  i.e.

$$M = \bigcup_{\alpha} U_\alpha$$

and for all  $\alpha, \beta \in I$  the corresponding **transition maps**

$$x_\beta \circ x_\alpha^{-1}|_{x_\alpha(U_\alpha \cap U_\beta)} : x_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^m \rightarrow \mathbb{R}^m$$

are  $r$ -times continuously differentiable.

A chart  $(U, x)$  on  $M$  is said to be **compatible** with a  $C^r$ -atlas  $\mathcal{A}$  if the union  $\mathcal{A} \cup \{(U, x)\}$  is a  $C^r$ -atlas. A  $C^r$ -atlas  $\hat{\mathcal{A}}$  is said to be **maximal** if it contains all the charts that are compatible with it. A maximal atlas  $\hat{\mathcal{A}}$  on  $M$  is also called a  $C^r$ -**structure** on  $M$ . The pair  $(M, \hat{\mathcal{A}})$  is said to be a  $C^r$ -**manifold**, or a **differentiable manifold** of class  $C^r$ , if  $M$  is a topological manifold and  $\hat{\mathcal{A}}$  is a  $C^r$ -structure on  $M$ . A differentiable manifold is said to be **smooth** if its transition maps are  $C^\infty$  and **real analytic** if they are  $C^\omega$ .

It should be noted that a given  $C^r$ -atlas  $\mathcal{A}$  on a topological manifold  $M$  determines a unique  $C^r$ -structure  $\hat{\mathcal{A}}$  on  $M$  containing  $\mathcal{A}$ . It simply consists of all charts compatible with  $\mathcal{A}$ .

**Example 2.3.** For the standard topological space  $(\mathbb{R}^m, \mathcal{T})$  we have the trivial  $C^\omega$ -atlas

$$\mathcal{A} = \{(\mathbb{R}^m, x) \mid x : p \mapsto p\}$$

inducing the standard  $C^\omega$ -structure  $\hat{\mathcal{A}}$  on  $\mathbb{R}^m$ .

**Example 2.4.** Let  $S^m$  denote the unit sphere in  $\mathbb{R}^{m+1}$  i.e.

$$S^m = \{p \in \mathbb{R}^{m+1} \mid p_1^2 + \cdots + p_{m+1}^2 = 1\}$$

equipped with the subset topology induced by the standard  $\mathcal{T}$  on  $\mathbb{R}^{m+1}$ . Let  $N$  be the north pole  $N = (1, 0) \in \mathbb{R} \times \mathbb{R}^m$  and  $S$  be the south pole  $S = (-1, 0)$  on  $S^m$ , respectively. Put  $U_N = S^m \setminus \{N\}$ ,  $U_S = S^m \setminus \{S\}$  and define  $x_N : U_N \rightarrow \mathbb{R}^m$ ,  $x_S : U_S \rightarrow \mathbb{R}^m$  by

$$x_N : (p_1, \dots, p_{m+1}) \mapsto \frac{1}{1 - p_1}(p_2, \dots, p_{m+1}),$$

$$x_S : (p_1, \dots, p_{m+1}) \mapsto \frac{1}{1 + p_1}(p_2, \dots, p_{m+1}).$$

Then the transition maps

$$x_S \circ x_N^{-1}, x_N \circ x_S^{-1} : \mathbb{R}^m \setminus \{0\} \rightarrow \mathbb{R}^m \setminus \{0\}$$

are given by

$$x \mapsto \frac{x}{|x|^2}$$

so  $\mathcal{A} = \{(U_N, x_N), (U_S, x_S)\}$  is a  $C^\omega$ -atlas on  $S^m$ . The  $C^\omega$ -manifold  $(S^m, \hat{\mathcal{A}})$  is called the  $m$ -dimensional **standard sphere**.

Another interesting example of a differentiable manifold is the  $m$ -dimensional real projective space  $\mathbb{R}P^m$ .

**Example 2.5.** On the set  $\mathbb{R}^{m+1} \setminus \{0\}$  we define the equivalence relation  $\equiv$  by

$$p \equiv q \text{ if and only if there exists a } \lambda \in \mathbb{R}^* \text{ such that } p = \lambda q.$$

Let  $\mathbb{R}P^m$  be the quotient space  $(\mathbb{R}^{m+1} \setminus \{0\}) / \equiv$  and

$$\pi : \mathbb{R}^{m+1} \setminus \{0\} \rightarrow \mathbb{R}P^m$$

be the natural projection mapping a point  $p \in \mathbb{R}^{m+1} \setminus \{0\}$  onto the equivalence class  $[p] \in \mathbb{R}P^m$  i.e. the line

$$[p] = \{\lambda p \in \mathbb{R}^{m+1} \mid \lambda \in \mathbb{R}^*\}$$

through the origin generated by  $p$ . Equip  $\mathbb{R}P^m$  with the quotient topology induced by  $\pi$  and  $\mathcal{T}$  on  $\mathbb{R}^{m+1}$ . For  $k \in \{1, \dots, m+1\}$  define the open subset

$$U_k = \{[p] \in \mathbb{R}P^m \mid p_k \neq 0\}$$

of  $\mathbb{R}P^m$  and the charts  $x_k : U_k \subset \mathbb{R}P^m \rightarrow \mathbb{R}^m$  by

$$x_k : [p] \mapsto \left( \frac{p_1}{p_k}, \dots, \frac{p_{k-1}}{p_k}, 1, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{m+1}}{p_k} \right).$$

If  $[p] = [q]$  then  $p = \lambda q$  for some  $\lambda \in \mathbb{R}^*$  so  $p_l/p_k = q_l/q_k$  for all  $l$ . This means that the map  $x_k$  is well defined for all  $k$ . The corresponding transition maps

$$x_k \circ x_l^{-1} |_{x_l(U_l \cap U_k)} : x_l(U_l \cap U_k) \subset \mathbb{R}^m \rightarrow \mathbb{R}^m$$

are given by

$$\left( \frac{p_1}{p_l}, \dots, \frac{p_{l-1}}{p_l}, 1, \frac{p_{l+1}}{p_l}, \dots, \frac{p_{m+1}}{p_l} \right) \mapsto \left( \frac{p_1}{p_k}, \dots, \frac{p_{k-1}}{p_k}, 1, \frac{p_{k+1}}{p_k}, \dots, \frac{p_{m+1}}{p_k} \right)$$

so the collection

$$\mathcal{A} = \{(U_k, x_k) \mid k = 1, \dots, m+1\}$$

is a  $C^\omega$ -atlas on  $\mathbb{R}P^m$ . The differentiable manifold  $(\mathbb{R}P^m, \hat{\mathcal{A}})$  is called the  $m$ -dimensional **real projective space**.

**Example 2.6.** Let  $\hat{\mathbb{C}}$  be the extended complex plane given by

$$\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$$

and put  $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ ,  $U_0 = \mathbb{C}$  and  $U_\infty = \hat{\mathbb{C}} \setminus \{0\}$ . Then define the local coordinates  $x_0 : U_0 \rightarrow \mathbb{C}$  and  $x_\infty : U_\infty \rightarrow \mathbb{C}$  on  $\hat{\mathbb{C}}$  by  $x_0 : z \mapsto z$  and  $x_\infty : w \mapsto 1/w$ , respectively. The corresponding transition maps

$$x_\infty \circ x_0^{-1}, x_0 \circ x_\infty^{-1} : \mathbb{C}^* \rightarrow \mathbb{C}^*$$

are both given by  $z \mapsto 1/z$  so  $\mathcal{A} = \{(U_0, x_0), (U_\infty, x_\infty)\}$  is a  $C^\omega$ -atlas on  $\hat{\mathbb{C}}$ . The real analytic manifold  $(\hat{\mathbb{C}}, \hat{\mathcal{A}})$  is called the **Riemann sphere**.

For the product of two differentiable manifolds we have the following important result.

**Proposition 2.7.** *Let  $(M_1, \hat{\mathcal{A}}_1)$  and  $(M_2, \hat{\mathcal{A}}_2)$  be two differentiable manifolds of class  $C^r$ . Let  $M = M_1 \times M_2$  be the product space with the product topology. Then there exists an atlas  $\mathcal{A}$  on  $M$  turning  $(M, \hat{\mathcal{A}})$  into a differentiable manifold of class  $C^r$  and the dimension of  $M$  satisfies*

$$\dim M = \dim M_1 + \dim M_2.$$

PROOF. See Exercise 2.1. □

The concept of a submanifold of a given differentiable manifold will play an important role as we go along and we shall be especially interested in the connection between the geometry of a submanifold and that of its ambient space.

**Definition 2.8.** Let  $m, n$  be positive integers with  $m \leq n$  and  $(N^n, \hat{\mathcal{A}}_N)$  be a  $C^r$ -manifold. A subset  $M$  of  $N$  is said to be a **submanifold** of  $N$  if for each point  $p \in M$  there exists a chart  $(U_p, x_p) \in \hat{\mathcal{A}}_N$  such that  $p \in U_p$  and  $x_p : U_p \subset N \rightarrow \mathbb{R}^m \times \mathbb{R}^{n-m}$  satisfies

$$x_p(U_p \cap M) = x_p(U_p) \cap (\mathbb{R}^m \times \{0\}).$$

The natural number  $(n - m)$  is called the **codimension** of  $M$  in  $N$ .

**Proposition 2.9.** *Let  $m, n$  be positive integers with  $m \leq n$  and  $(N^n, \hat{\mathcal{A}}_N)$  be a  $C^r$ -manifold. Let  $M$  be a submanifold of  $N$  equipped with the subset topology and  $\pi : \mathbb{R}^m \times \mathbb{R}^{n-m} \rightarrow \mathbb{R}^m$  be the natural projection onto the first factor. Then*

$$\mathcal{A}_M = \{(U_p \cap M, (\pi \circ x_p)|_{U_p \cap M}) \mid p \in M\}$$

is a  $C^r$ -atlas for  $M$ . Hence the pair  $(M, \hat{\mathcal{A}}_M)$  is an  $m$ -dimensional  $C^r$ -manifold. The differentiable structure  $\hat{\mathcal{A}}_M$  on  $M$  is called the **induced structure** of  $\hat{\mathcal{A}}_N$ .

PROOF. See Exercise 2.2.  $\square$

**Remark 2.10.** Our next aim is to prove Theorem 2.14 which is a useful tool for the construction of submanifolds of  $\mathbb{R}^m$ . For this we use the classical inverse function theorem stated below. Note that if

$$F : U \rightarrow \mathbb{R}^n$$

is a differentiable map defined on an open subset  $U$  of  $\mathbb{R}^m$  then its differential  $dF_p : \mathbb{R}^m \rightarrow \mathbb{R}^n$  at the point  $p \in U$  is a linear map given by the  $n \times m$  matrix

$$dF_p = \begin{pmatrix} \partial F_1 / \partial x_1(p) & \dots & \partial F_1 / \partial x_m(p) \\ \vdots & & \vdots \\ \partial F_n / \partial x_1(p) & \dots & \partial F_n / \partial x_m(p) \end{pmatrix}.$$

If  $\gamma : \mathbb{R} \rightarrow U$  is a curve in  $U$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v \in \mathbb{R}^m$  then the composition  $F \circ \gamma : \mathbb{R} \rightarrow \mathbb{R}^n$  is a curve in  $\mathbb{R}^n$  and according to the chain rule we have

$$dF_p \cdot v = \frac{d}{ds}(F \circ \gamma(s))|_{s=0}.$$

This is the tangent vector of the curve  $F \circ \gamma$  at  $F(p) \in \mathbb{R}^n$ .

**Hence the differential  $dF_p$  can be seen as a linear map that maps tangent vectors at  $p \in U$  to tangent vectors at the image  $F(p) \in \mathbb{R}^n$ . This will later be generalized to the manifold setting.**

**Fact 2.11** (The Inverse Function Theorem). *Let  $U$  be an open subset of  $\mathbb{R}^m$  and  $F : U \rightarrow \mathbb{R}^m$  be a  $C^r$ -map. If  $p \in U$  and the differential*

$$dF_p : \mathbb{R}^m \rightarrow \mathbb{R}^m$$

*of  $F$  at  $p$  is invertible then there exist open neighbourhoods  $U_p$  around  $p$  and  $U_q$  around  $q = F(p)$  such that  $\hat{F} = F|_{U_p} : U_p \rightarrow U_q$  is bijective and the inverse  $(\hat{F})^{-1} : U_q \rightarrow U_p$  is a  $C^r$ -map. The differential  $(d\hat{F}^{-1})_q$  of  $\hat{F}^{-1}$  at  $q$  satisfies*

$$(d\hat{F}^{-1})_q = (dF_p)^{-1}$$

*i.e. it is the inverse of the differential  $dF_p$  of  $F$  at  $p$ .*

Before stating the implicit function theorem we remind the reader of the following notions.

**Definition 2.12.** Let  $m, n$  be positive natural numbers,  $U$  be an open subset of  $\mathbb{R}^m$  and  $F : U \rightarrow \mathbb{R}^n$  be a  $C^r$ -map. A point  $p \in U$  is said to be **critical** for  $F$  if the differential

$$dF_p : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is not of full rank, and **regular** if it is not critical. A point  $q \in F(U)$  is said to be a **regular value** of  $F$  if every point in the pre-image  $F^{-1}(\{q\})$  of  $q$  is regular and a **critical value** otherwise.

**Remark 2.13.** Note that if  $m, n$  are positive integers with  $m \geq n$  then  $p \in U$  is a regular point of

$$F = (F_1, \dots, F_n) : U \rightarrow \mathbb{R}^n$$

if and only if the gradients  $\text{grad}F_1, \dots, \text{grad}F_n$  of the coordinate functions  $F_1, \dots, F_n : U \rightarrow \mathbb{R}$  are linearly independent at  $p$ , or equivalently, the differential  $dF_p$  of  $F$  at  $p$  satisfies the following condition

$$\det(dF_p \cdot (dF_p)^t) \neq 0.$$

**Theorem 2.14** (The Implicit Function Theorem). *Let  $m, n$  be positive integers with  $m > n$  and  $F : U \rightarrow \mathbb{R}^n$  be a  $C^r$ -map from an open subset  $U$  of  $\mathbb{R}^m$ . If  $q \in F(U)$  is a regular value of  $F$  then the pre-image  $F^{-1}(\{q\})$  of  $q$  is an  $(m - n)$ -dimensional submanifold of  $\mathbb{R}^m$  of class  $C^r$ .*

PROOF. Let  $p$  be an element of  $F^{-1}(\{q\})$  and  $K_p$  be the kernel of the differential  $dF_p$  i.e. the  $(m - n)$ -dimensional subspace of  $\mathbb{R}^m$  given by  $K_p = \{v \in \mathbb{R}^m \mid dF_p \cdot v = 0\}$ . Let  $\pi_p : \mathbb{R}^m \rightarrow \mathbb{R}^{m-n}$  be a linear map such that  $\pi_p|_{K_p} : K_p \rightarrow \mathbb{R}^{m-n}$  is bijective,  $\pi_p|_{K_p^\perp} = 0$  and define the map  $G_p : U \rightarrow \mathbb{R}^n \times \mathbb{R}^{m-n}$  by

$$G_p : x \mapsto (F(x), \pi_p(x)).$$

Then the differential  $(dG_p)_p : \mathbb{R}^m \rightarrow \mathbb{R}^m$  of  $G_p$ , with respect to the decompositions  $\mathbb{R}^m = K_p^\perp \oplus K_p$  and  $\mathbb{R}^m = \mathbb{R}^n \oplus \mathbb{R}^{m-n}$ , is given by

$$(dG_p)_p = \begin{pmatrix} dF_p|_{K_p^\perp} & 0 \\ 0 & \pi_p \end{pmatrix},$$

hence bijective. It now follows from the inverse function theorem that there exist open neighbourhoods  $V_p$  around  $p$  and  $W_p$  around  $G_p(p)$  such that  $\hat{G}_p = G_p|_{V_p} : V_p \rightarrow W_p$  is bijective, the inverse  $\hat{G}_p^{-1} : W_p \rightarrow V_p$  is  $C^r$ ,  $d(\hat{G}_p^{-1})_{G_p(p)} = (dG_p)_p^{-1}$  and  $d(\hat{G}_p^{-1})_y$  is bijective for all  $y \in W_p$ . Now put  $\tilde{U}_p = F^{-1}(\{q\}) \cap V_p$  then

$$\tilde{U}_p = \hat{G}_p^{-1}(\{q\} \times \mathbb{R}^{m-n}) \cap W_p$$

so if  $\pi : \mathbb{R}^n \times \mathbb{R}^{m-n} \rightarrow \mathbb{R}^{m-n}$  is the natural projection onto the second factor, then the map

$$\tilde{x}_p = \pi \circ G_p|_{\tilde{U}_p} : \tilde{U}_p \rightarrow (\{q\} \times \mathbb{R}^{m-n}) \cap W_p \rightarrow \mathbb{R}^{m-n}$$

is a chart on the open neighbourhood  $\tilde{U}_p$  of  $p$ . The point  $q \in F(U)$  is a regular value so the set

$$\mathcal{B} = \{(\tilde{U}_p, \tilde{x}_p) \mid p \in F^{-1}(\{q\})\}$$

is a  $C^r$ -atlas for  $F^{-1}(\{q\})$ .  $\square$

Employing the implicit function theorem, we obtain the following interesting examples of the  $m$ -dimensional sphere  $S^m$  and its tangent bundle  $TS^m$  as differentiable submanifolds of  $\mathbb{R}^{m+1}$  and  $\mathbb{R}^{2m+2}$ , respectively.

**Example 2.15.** Let  $F : \mathbb{R}^{m+1} \rightarrow \mathbb{R}$  be the  $C^\omega$ -map given by

$$F : (p_1, \dots, p_{m+1}) \mapsto p_1^2 + \dots + p_{m+1}^2.$$

Then the differential  $dF_p$  of  $F$  at  $p$  is given by  $dF_p = 2p$ , so

$$dF_p \cdot (dF_p)^t = 4|p|^2 \in \mathbb{R}.$$

This means that  $1 \in \mathbb{R}$  is a regular value for  $F$  so the fibre

$$S^m = \{p \in \mathbb{R}^{m+1} \mid |p|^2 = 1\} = F^{-1}(\{1\})$$

of  $F$  is an  $m$ -dimensional submanifold of  $\mathbb{R}^{m+1}$ . This is the standard  $m$ -dimensional sphere introduced in Example 2.4.

**Example 2.16.** Let  $F : \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \rightarrow \mathbb{R}^2$  be the  $C^\omega$ -map defined by  $F : (p, v) \mapsto (|p|^2 - 1)/2, \langle p, v \rangle$ . Then the differential  $dF_{(p,v)}$  of  $F$  at  $(p, v)$  satisfies

$$dF_{(p,v)} = \begin{pmatrix} p & 0 \\ v & p \end{pmatrix}.$$

A simple calculation shows that

$$\det(dF \cdot (dF)^t) = \det \begin{pmatrix} |p|^2 & \langle p, v \rangle \\ \langle p, v \rangle & |v|^2 + |p|^2 \end{pmatrix} = 1 + |v|^2 > 0$$

on  $F^{-1}(\{0\})$ . This means that

$$F^{-1}(\{0\}) = \{(p, v) \in \mathbb{R}^{m+1} \times \mathbb{R}^{m+1} \mid |p|^2 = 1 \text{ and } \langle p, v \rangle = 0\},$$

which we denote by  $TS^m$ , is a  $2m$ -dimensional submanifold of  $\mathbb{R}^{2m+2}$ . We will later see that  $TS^m$  is what is called the tangent bundle of the  $m$ -dimensional sphere.

We now apply the implicit function theorem to construct the important orthogonal group  $\mathbf{O}(m)$  as a submanifold of the set of the real vector space of  $m \times m$  matrices  $\mathbb{R}^{m \times m}$ .

**Example 2.17.** Let  $\text{Sym}(\mathbb{R}^m)$  be the  $m(m+1)/2$  dimensional linear subspace of  $\mathbb{R}^{m \times m}$  consisting of all symmetric  $m \times m$  matrices

$$\text{Sym}(\mathbb{R}^m) = \{y \in \mathbb{R}^{m \times m} \mid y^t = y\}.$$

Let  $F : \mathbb{R}^{m \times m} \rightarrow \text{Sym}(\mathbb{R}^m)$  be the map defined by

$$F : x \mapsto x^t x.$$

If  $\gamma : I \rightarrow \mathbb{R}^{m \times m}$  is a curve in  $\mathbb{R}^{m \times m}$  then

$$\frac{d}{ds}(F \circ \gamma(s)) = \dot{\gamma}(s)^t \gamma(s) + \gamma(s)^t \dot{\gamma}(s),$$

so the differential  $dF_x$  of  $F$  at  $x \in \mathbb{R}^{m \times m}$  satisfies

$$dF_x : X \mapsto X^t x + x^t X.$$

This means that for an arbitrary element  $p$  in

$$\mathbf{O}(m) = F^{-1}(\{e\}) = \{p \in \mathbb{R}^{m \times m} \mid p^t p = e\}$$

and  $Y \in \text{Sym}(\mathbb{R}^m)$  we have  $dF_p(pY/2) = Y$ . Hence the differential  $dF_p$  is surjective, so the identity matrix  $e \in \text{Sym}(\mathbb{R}^m)$  is a regular value for  $F$ . Following the implicit function theorem  $\mathbf{O}(m)$  is a submanifold of  $\mathbb{R}^{m \times m}$  of dimension  $m(m-1)/2$ . The set  $\mathbf{O}(m)$  is the well known **orthogonal group**.

The concept of a differentiable map  $U \rightarrow \mathbb{R}^n$ , defined on an open subset of  $\mathbb{R}^m$ , can be generalized to mappings between manifolds. We will see that the most important properties of these objects in the classical case are also valid in the manifold setting.

**Definition 2.18.** Let  $(M^m, \hat{\mathcal{A}}_M)$  and  $(N^n, \hat{\mathcal{A}}_N)$  be  $C^r$ -manifolds. A map  $\phi : M \rightarrow N$  is said to be **differentiable** of class  $C^r$  if for all charts  $(U, x) \in \hat{\mathcal{A}}_M$  and  $(V, y) \in \hat{\mathcal{A}}_N$  the map

$$y \circ \phi \circ x^{-1}|_{x(U \cap \phi^{-1}(V))} : x(U \cap \phi^{-1}(V)) \subset \mathbb{R}^m \rightarrow \mathbb{R}^n$$

is of class  $C^r$ . A differentiable map  $\gamma : I \rightarrow M$  defined on an open interval of  $\mathbb{R}$  is called a differentiable **curve** in  $M$ . A differentiable map  $f : M \rightarrow \mathbb{R}$  with values in  $\mathbb{R}$  is called a differentiable **function** on  $M$ . The set of smooth functions defined on  $M$  is denoted by  $C^\infty(M)$ .

It is an easy exercise, using Definition 2.18, to prove the following result concerning the composition of differentiable maps between manifolds.



**Proposition 2.19.** *Let  $(M_1, \hat{\mathcal{A}}_1), (M_2, \hat{\mathcal{A}}_2), (M_3, \hat{\mathcal{A}}_3)$  be  $C^r$ -manifolds and  $\phi : (M_1, \hat{\mathcal{A}}_1) \rightarrow (M_2, \hat{\mathcal{A}}_2), \psi : (M_2, \hat{\mathcal{A}}_2) \rightarrow (M_3, \hat{\mathcal{A}}_3)$  be differentiable maps of class  $C^r$ . Then the composition  $\psi \circ \phi : (M_1, \hat{\mathcal{A}}_1) \rightarrow (M_3, \hat{\mathcal{A}}_3)$  is a differentiable map of class  $C^r$ .*

PROOF. See Exercise 2.5. □

**Definition 2.20.** Two manifolds  $(M, \hat{\mathcal{A}}_M)$  and  $(N, \hat{\mathcal{A}}_N)$  of class  $C^r$  are said to be **diffeomorphic** if there exists a bijective  $C^r$ -map  $\phi : M \rightarrow N$  such that the inverse  $\phi^{-1} : N \rightarrow M$  is of class  $C^r$ . In that case the map  $\phi$  is called a **diffeomorphism** between  $(M, \hat{\mathcal{A}}_M)$  and  $(N, \hat{\mathcal{A}}_N)$ .

It can be shown that the 2-dimensional sphere  $S^2$  in  $\mathbb{R}^3$  and the Riemann sphere  $\hat{\mathbb{C}}$  are diffeomorphic, see Exercise 2.7.

**Definition 2.21.** For a differentiable manifold  $(M, \hat{\mathcal{A}})$  we denote by  $\mathcal{D}(M)$  the set of all its diffeomorphisms. If  $\phi, \psi \in \mathcal{D}(M)$  then it is clear that the composition  $\psi \circ \phi$  and the inverse  $\phi^{-1}$  are also diffeomorphisms. The pair  $(\mathcal{D}(M), \circ)$  is called the **diffeomorphism group** of  $(M, \hat{\mathcal{A}})$ . The operation is clearly associative and the identity map is its neutral element.

**Definition 2.22.** Two  $C^r$ -structures  $\hat{\mathcal{A}}_1$  and  $\hat{\mathcal{A}}_2$  on the same topological manifold  $M$  are said to be **different** if the identity map  $\text{id}_M : (M, \hat{\mathcal{A}}_1) \rightarrow (M, \hat{\mathcal{A}}_2)$  is not a diffeomorphism.

It can be seen that even the real line  $\mathbb{R}$  carries different differentiable structures, see Exercise 2.6.

**Deep Result 2.23.** *Let  $(M, \hat{\mathcal{A}}_M), (N, \hat{\mathcal{A}}_N)$  be differentiable manifolds of class  $C^r$  of the same dimension  $m$ . If  $M$  and  $N$  are homeomorphic as topological spaces and  $m \leq 3$  then  $(M, \hat{\mathcal{A}}_M)$  and  $(N, \hat{\mathcal{A}}_N)$  are diffeomorphic.*

The following remarkable result was proven by John Milnor in his famous paper: *Differentiable structures on spheres*, Amer. J. Math. **81** (1959), 962-972.

**Deep Result 2.24.** *The 7-dimensional sphere  $S^7$  has exactly 28 different differentiable structures.*

The next very useful proposition generalizes a classical result from real analysis of several variables.

**Proposition 2.25.** *Let  $(N_1, \hat{\mathcal{A}}_1)$  and  $(N_2, \hat{\mathcal{A}}_2)$  be two differentiable manifolds of class  $C^r$  and  $M_1, M_2$  be submanifolds of  $N_1$  and  $N_2$ , respectively. If  $\phi : N_1 \rightarrow N_2$  is a differentiable map of class  $C^r$  such that  $\phi(M_1)$  is contained in  $M_2$  then the restriction  $\phi|_{M_1} : M_1 \rightarrow M_2$  is differentiable of class  $C^r$ .*

PROOF. See Exercise 2.8.  $\square$

**Example 2.26.** The result of Proposition 2.25 implies that the following maps are all smooth.

- (i)  $\phi_1 : \mathbb{R}^1 \rightarrow S^1 \subset \mathbb{C}$ ,  $\phi_1 : t \mapsto e^{it}$ ,
- (ii)  $\phi_2 : \mathbb{R}^{m+1} \setminus \{0\} \rightarrow S^m \subset \mathbb{R}^m$ ,  $\phi_2 : x \mapsto x/|x|$ ,
- (iii)  $\phi_3 : S^2 \subset \mathbb{R}^3 \rightarrow S^3 \subset \mathbb{R}^4$ ,  $\phi_3 : (x, y, z) \mapsto (x, y, z, 0)$ ,
- (iv)  $\phi_4 : S^3 \subset \mathbb{C}^2 \rightarrow S^2 \subset \mathbb{C} \times \mathbb{R}$ ,  $\phi_4 : (z_1, z_2) \mapsto (2z_1\bar{z}_2, |z_1|^2 - |z_2|^2)$ ,
- (v)  $\phi_5 : \mathbb{R}^{m+1} \setminus \{0\} \rightarrow \mathbb{R}P^m$ ,  $\phi_5 : x \mapsto [x]$ ,
- (vi)  $\phi_6 : S^m \rightarrow \mathbb{R}P^m$ ,  $\phi_6 : x \mapsto [x]$ .

In differential geometry we are specially interested in differentiable manifolds carrying a group structure compatible with their differentiable structure. Such manifolds are named after the famous mathematician Sophus Lie (1842-1899) and will play an important role throughout this work.

**Definition 2.27.** A **Lie group** is a smooth manifold  $G$  with a group structure  $\cdot$  such that the map  $\rho : G \times G \rightarrow G$  with

$$\rho : (p, q) \mapsto p \cdot q^{-1}$$

is smooth. For an element  $p$  in  $G$  the **left translation** by  $p$  is the map  $L_p : G \rightarrow G$  defined by  $L_p : q \mapsto p \cdot q$ .

**Example 2.28.** Let  $(\mathbb{R}^m, +, \cdot)$  be the  $m$ -dimensional vector space equipped with its standard differential structure. Then  $(\mathbb{R}^m, +)$  with  $\rho : \mathbb{R}^m \times \mathbb{R}^m \rightarrow \mathbb{R}^m$  given by

$$\rho : (p, q) \mapsto p - q$$

is a Lie group.

**Corollary 2.29.** *Let  $G$  be a Lie group and  $p$  be an element of  $G$ . Then the left translation  $L_p : G \rightarrow G$  is a smooth diffeomorphism.*

PROOF. See Exercise 2.10  $\square$

**Proposition 2.30.** *Let  $(G, \cdot)$  be a Lie group and  $K$  be a submanifold of  $G$  which is a subgroup. Then  $(K, \cdot)$  is a Lie group.*

PROOF. The statement is a direct consequence of Definition 2.27 and Proposition 2.25.  $\square$

The set of non-zero complex numbers  $(\mathbb{C}^*, \cdot)$ , together with the standard multiplication, form a Lie group. The unit circle  $(S^1, \cdot)$  is an interesting compact Lie subgroup of  $(\mathbb{C}^*, \cdot)$ . Another subgroup is the set of the non-zero real numbers  $(\mathbb{R}^*, \cdot)$  containing the positive real numbers  $(\mathbb{R}^+, \cdot)$  as a subgroup.

The Lie group of non-zero complex numbers  $(\mathbb{C}^*, \cdot)$  has a well-known linear representation  $\rho : \mathbb{C}^* \rightarrow \mathbb{R}^{2 \times 2}$  given by

$$\rho : a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}.$$

This is obviously injective and it respects the standard multiplicative structures of  $\mathbb{C}^*$  and  $\mathbb{R}^{2 \times 2}$  i.e.

$$\begin{aligned} \rho(a + ib) \cdot \rho(x + iy) &= \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \\ &= \begin{pmatrix} ax - by & ay + bx \\ -(ay + bx) & ax - by \end{pmatrix} \\ &= \rho((ax - by) + i(ay + bx)) \\ &= \rho((a + ib)(x + iy)). \end{aligned}$$

As an introduction to Example 2.31 we now play the same game in the complex case: Let  $\sigma : \mathbb{C}^2 \rightarrow \mathbb{C}^{2 \times 2}$  be the complex linear map given by

$$\sigma : (z, w) \mapsto \begin{pmatrix} z & w \\ -\bar{w} & \bar{z} \end{pmatrix}.$$

Then an easy calculation shows that the following is true

$$\begin{aligned} \sigma(z_1, w_1) \cdot \sigma(z_2, w_2) &= \begin{pmatrix} z_1 & w_1 \\ -\bar{w}_1 & \bar{z}_1 \end{pmatrix} \cdot \begin{pmatrix} z_2 & w_2 \\ -\bar{w}_2 & \bar{z}_2 \end{pmatrix} \\ &= \begin{pmatrix} z_1 z_2 - w_1 \bar{w}_2 & z_1 w_2 + w_1 \bar{z}_2 \\ -\bar{z}_1 \bar{w}_2 - \bar{w}_1 z_2 & \bar{z}_1 \bar{z}_2 - \bar{w}_1 w_2 \end{pmatrix} \\ &= \sigma(z_1 z_2 - w_1 \bar{w}_2, z_1 w_2 + w_1 \bar{z}_2). \end{aligned}$$

**Example 2.31.** Let  $\mathbb{H}$  be the set of **quaternions** given by

$$\mathbb{H} = \{z + wj \mid z, w \in \mathbb{C}\}.$$

We equip  $\mathbb{H}$  with an addition, a multiplication and the conjugation satisfying

- (i)  $\overline{(z + wj)} = \bar{z} - wj$ ,
- (ii)  $(z_1 + w_1j) + (z_2 + w_2j) = (z_1 + z_2) + (w_1 + w_2)j$ ,
- (iii)  $(z_1 + w_1j) \cdot (z_2 + w_2j) = (z_1 z_2 - w_1 \bar{w}_2) + (z_1 w_2 + w_1 \bar{z}_2)j$ .

These extend the standard operations on  $\mathbb{C}$  as a subset of  $\mathbb{H}$ . It is easily seen that the non-zero quaternions  $(\mathbb{H}^*, \cdot)$  form a Lie group. On  $\mathbb{H}$  we define a scalar product

$$\mathbb{H} \times \mathbb{H} \rightarrow \mathbb{H}, \quad (p, q) \mapsto p \cdot \bar{q}$$

and a real valued norm given by  $|p|^2 = p \cdot \bar{p}$ . Then the 3-dimensional unit sphere  $S^3$  in  $\mathbb{H} \cong \mathbb{R}^4$ , with the restricted multiplication, forms a compact Lie subgroup  $(S^3, \cdot)$  of  $(\mathbb{H}^*, \cdot)$ . They are both non-abelian.

We shall now introduce some of the classical real and complex matrix Lie groups. As a reference on this topic we recommend the wonderful book: A. W. Knap, *Lie Groups Beyond an Introduction*, Birkhäuser (2002).

**Example 2.32.** Let  $\text{Nil}^3$  be the subset of  $\mathbb{R}^{3 \times 3}$  given by

$$\text{Nil}^3 = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3} \mid x, y, z \in \mathbb{R} \right\}.$$

Then  $\text{Nil}^3$  has a natural differentiable structure determined by the global coordinates  $\phi : \text{Nil}^3 \rightarrow \mathbb{R}^3$  with

$$\phi : \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z).$$

It is easily seen that if  $*$  is the standard matrix multiplication, then  $(\text{Nil}^3, *)$  is a Lie group.

**Example 2.33.** Let  $\text{Sol}^3$  be the subset of  $\mathbb{R}^{3 \times 3}$  given by

$$\text{Sol}^3 = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathbb{R}^{3 \times 3} \mid x, y, z \in \mathbb{R} \right\}.$$

Then  $\text{Sol}^3$  has a natural differentiable structure determined by the global coordinates  $\phi : \text{Sol}^3 \rightarrow \mathbb{R}^3$  with

$$\phi : \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z).$$

It is easily seen that if  $*$  is the standard matrix multiplication, then  $(\text{Sol}^3, *)$  is a Lie group.

**Example 2.34.** The set of invertible real  $m \times m$  matrices

$$\text{GL}_m(\mathbb{R}) = \{A \in \mathbb{R}^{m \times m} \mid \det A \neq 0\}$$

equipped with the standard matrix multiplication has the structure of a Lie group. It is called the **real general linear group** and its neutral element  $e$  is the identity matrix. The subset  $\mathbf{GL}_m(\mathbb{R})$  of  $\mathbb{R}^{m \times m}$  is open so  $\dim \mathbf{GL}_m(\mathbb{R}) = m^2$ .

As a subgroup of  $\mathbf{GL}_m(\mathbb{R})$  we have the **real special linear group**  $\mathbf{SL}_m(\mathbb{R})$  given by

$$\mathbf{SL}_m(\mathbb{R}) = \{A \in \mathbb{R}^{m \times m} \mid \det A = 1\}.$$

We will show in Example 3.10 that the dimension of the submanifold  $\mathbf{SL}_m(\mathbb{R})$  of  $\mathbb{R}^{m \times m}$  is  $m^2 - 1$ .

Another subgroup of  $\mathbf{GL}_m(\mathbb{R})$  is the **orthogonal group**

$$\mathbf{O}(m) = \{A \in \mathbb{R}^{m \times m} \mid A^t A = e\}.$$

As we have already seen in Example 2.17 the dimension of  $\mathbf{O}(m)$  is  $m(m-1)/2$ .

As a subgroup of  $\mathbf{O}(m)$  and  $\mathbf{SL}_m(\mathbb{R})$  we have the **special orthogonal group**  $\mathbf{SO}(m)$  which is defined as

$$\mathbf{SO}(m) = \mathbf{O}(m) \cap \mathbf{SL}_m(\mathbb{R}).$$

It can be shown that  $\mathbf{O}(m)$  is diffeomorphic to  $\mathbf{SO}(m) \times \mathbf{O}(1)$ , see Exercise 2.9. Note that  $\mathbf{O}(1) = \{\pm 1\}$  so  $\mathbf{O}(m)$  can be seen as double cover of  $\mathbf{SO}(m)$ . This means that

$$\dim \mathbf{SO}(m) = \dim \mathbf{O}(m) = m(m-1)/2.$$

**Example 2.35.** The set of invertible complex  $m \times m$  matrices

$$\mathbf{GL}_m(\mathbb{C}) = \{A \in \mathbb{C}^{m \times m} \mid \det A \neq 0\}$$

equipped with the standard matrix multiplication has the structure of a Lie group. It is called the **complex general linear group** and its neutral element  $e$  is the identity matrix. The subset  $\mathbf{GL}_m(\mathbb{C})$  of  $\mathbb{C}^{m \times m}$  is open so  $\dim(\mathbf{GL}_m(\mathbb{C})) = 2m^2$ .

As a subgroup of  $\mathbf{GL}_m(\mathbb{C})$  we have the **complex special linear group**  $\mathbf{SL}_m(\mathbb{C})$  given by

$$\mathbf{SL}_m(\mathbb{C}) = \{A \in \mathbb{C}^{m \times m} \mid \det A = 1\}.$$

The dimension of the submanifold  $\mathbf{SL}_m(\mathbb{C})$  of  $\mathbb{C}^{m \times m}$  is  $2(m^2 - 1)$ .

Another subgroup of  $\mathbf{GL}_m(\mathbb{C})$  is the **unitary group**  $\mathbf{U}(m)$  given by

$$\mathbf{U}(m) = \{A \in \mathbb{C}^{m \times m} \mid \bar{A}^t A = e\}.$$

Calculations similar to those for the orthogonal group show that the dimension of  $\mathbf{U}(m)$  is  $m^2$ .

As a subgroup of  $\mathbf{U}(m)$  and  $\mathbf{SL}_m(\mathbb{C})$  we have the **special unitary group**  $\mathbf{SU}(m)$  which is defined as

$$\mathbf{SU}(m) = \mathbf{U}(m) \cap \mathbf{SL}_m(\mathbb{C}).$$

It can be shown that  $\mathbf{U}(1)$  is diffeomorphic to the circle  $S^1$  and that  $\mathbf{U}(m)$  is diffeomorphic to  $\mathbf{SU}(m) \times \mathbf{U}(1)$ , see Exercise 2.9. This means that  $\dim \mathbf{SU}(m) = m^2 - 1$ .

**For the rest of this work we shall assume, when not stating otherwise, that our manifolds and maps are smooth i.e. in the  $C^\infty$ -category.**

## Exercises

**Exercise 2.1.** Find a proof of Proposition 2.7.

**Exercise 2.2.** Find a proof of Proposition 2.9.

**Exercise 2.3.** Let  $S^1$  be the unit circle in the complex plane  $\mathbb{C}$  given by  $S^1 = \{z \in \mathbb{C} \mid |z|^2 = 1\}$ . Use the maps  $x : \mathbb{C} \setminus \{i\} \rightarrow \mathbb{C}$  and  $y : \mathbb{C} \setminus \{-i\} \rightarrow \mathbb{C}$  with

$$x : z \mapsto \frac{i+z}{1+iz}, \quad y : z \mapsto \frac{1+iz}{i+z}$$

to show that  $S^1$  is a 1-dimensional submanifold of  $\mathbb{C} \cong \mathbb{R}^2$ .

**Exercise 2.4.** Use the implicit function theorem to show that the  $m$ -dimensional **torus**

$$T^m = \{z \in \mathbb{C}^m \mid |z_1|^2 = \cdots = |z_m|^2 = 1\}$$

is a differentiable submanifold of  $\mathbb{C}^m \cong \mathbb{R}^{2m}$ .

**Exercise 2.5.** Find a proof of Proposition 2.19.

**Exercise 2.6.** Equip the real line  $\mathbb{R}$  with the standard topology and for each odd integer  $k \in \mathbb{Z}^+$  let  $\hat{\mathcal{A}}_k$  be the  $C^\omega$ -structure defined on  $\mathbb{R}$  by the atlas

$$\mathcal{A}_k = \{(\mathbb{R}, x_k) \mid x_k : p \mapsto p^k\}.$$

Show that the differentiable structures  $\hat{\mathcal{A}}_k$  are all different but that the differentiable manifolds  $(\mathbb{R}, \hat{\mathcal{A}}_k)$  are all diffeomorphic.

**Exercise 2.7.** Prove that the 2-dimensional sphere  $S^2$  as a differentiable submanifold of the standard  $\mathbb{R}^3$  and the Riemann sphere  $\hat{\mathbb{C}}$  are diffeomorphic.

**Exercise 2.8.** Find a proof of Proposition 2.25.

**Exercise 2.9.** Let the spheres  $S^1$ ,  $S^3$  and the Lie groups  $\mathbf{SO}(n)$ ,  $\mathbf{O}(n)$ ,  $\mathbf{SU}(n)$ ,  $\mathbf{U}(n)$  be equipped with their standard differentiable structures. Use Proposition 2.25 to prove the following diffeomorphisms

$$S^1 \cong \mathbf{SO}(2), \quad S^3 \cong \mathbf{SU}(2),$$

$$\mathbf{SO}(n) \times \mathbf{O}(1) \cong \mathbf{O}(n), \quad \mathbf{SU}(n) \times \mathbf{U}(1) \cong \mathbf{U}(n).$$

**Exercise 2.10.** Find a proof of Corollary 2.29.

**Exercise 2.11.** Let  $(G, *)$  and  $(H, \cdot)$  be two Lie groups. Prove that the product manifold  $G \times H$  has the structure of a Lie group.





## The Tangent Space

In this chapter we introduce the notion of the tangent space  $T_pM$  of a differentiable manifold  $M$  at a point  $p \in M$ . This is a vector space of the same dimension as  $M$ . We start by studying the standard  $\mathbb{R}^m$  and show how a tangent vector  $v$  at a point  $p \in \mathbb{R}^m$  can be interpreted as a first order linear differential operator, annihilating constants, when acting on real-valued functions locally defined around  $p$ .

Let  $\mathbb{R}^m$  be the  $m$ -dimensional real vector space with the standard differentiable structure. If  $p$  is a point in  $\mathbb{R}^m$  and  $\gamma : I \rightarrow \mathbb{R}^m$  is a  $C^1$ -curve such that  $\gamma(0) = p$  then the **tangent vector**

$$\dot{\gamma}(0) = \lim_{t \rightarrow 0} \frac{\gamma(t) - \gamma(0)}{t}$$

of  $\gamma$  at  $p$  is an element of  $\mathbb{R}^m$ . Conversely, for an arbitrary element  $v$  of  $\mathbb{R}^m$  we can easily find a curve  $\gamma : I \rightarrow \mathbb{R}^m$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . One example is given by

$$\gamma : t \mapsto p + t \cdot v.$$

This shows that the **tangent space**, i.e. the set of tangent vectors, at the point  $p \in \mathbb{R}^m$  can be identified with  $\mathbb{R}^m$ .

We shall now describe how first order differential operators annihilating constants can be interpreted as tangent vectors. For a point  $p$  in  $\mathbb{R}^m$  we denote by  $\varepsilon(p)$  the set of differentiable real-valued functions defined locally around  $p$ . Then it is well known from multi-variable analysis that if  $v \in \mathbb{R}^m$  and  $f \in \varepsilon(p)$  then the **directional derivative**  $\partial_v f$  of  $f$  at  $p$  in the direction of  $v$  is given by

$$\partial_v f = \lim_{t \rightarrow 0} \frac{f(p + tv) - f(p)}{t}.$$

Furthermore the operator  $\partial$  has the following properties

$$\begin{aligned} \partial_v(\lambda \cdot f + \mu \cdot g) &= \lambda \cdot \partial_v f + \mu \cdot \partial_v g, \\ \partial_v(f \cdot g) &= \partial_v f \cdot g(p) + f(p) \cdot \partial_v g, \\ \partial_{(\lambda \cdot v + \mu \cdot w)} f &= \lambda \cdot \partial_v f + \mu \cdot \partial_w f \end{aligned}$$

for all  $\lambda, \mu \in \mathbb{R}$ ,  $v, w \in \mathbb{R}^m$  and  $f, g \in \varepsilon(p)$ .

**Definition 3.1.** For a point  $p$  in  $\mathbb{R}^m$  let  $T_p\mathbb{R}^m$  be the set of first order linear differential operators at  $p$  annihilating constants i.e. the set of mappings  $\alpha : \varepsilon(p) \rightarrow \mathbb{R}$  such that

- (i)  $\alpha(\lambda \cdot f + \mu \cdot g) = \lambda \cdot \alpha(f) + \mu \cdot \alpha(g)$ ,
- (ii)  $\alpha(f \cdot g) = \alpha(f) \cdot g(p) + f(p) \cdot \alpha(g)$

for all  $\lambda, \mu \in \mathbb{R}$  and  $f, g \in \varepsilon(p)$ .

The set of differential operators  $T_p\mathbb{R}^m$  carries the structure of a real vector space. This is given by the addition  $+$  and the multiplication  $\cdot$  by real numbers satisfying

$$\begin{aligned}(\alpha + \beta)(f) &= \alpha(f) + \beta(f), \\ (\lambda \cdot \alpha)(f) &= \lambda \cdot \alpha(f)\end{aligned}$$

for all  $\alpha, \beta \in T_p\mathbb{R}^m$ ,  $f \in \varepsilon(p)$  and  $\lambda \in \mathbb{R}$ .

The above mentioned properties of the operator  $\partial$  show that we have a well defined linear map  $\Phi : \mathbb{R}^m \rightarrow T_p\mathbb{R}^m$  given by

$$\Phi : v \mapsto \partial_v.$$

**Theorem 3.2.** For a point  $p$  in  $\mathbb{R}^m$  the linear map  $\Phi : \mathbb{R}^m \rightarrow T_p\mathbb{R}^m$  defined by  $\Phi : v \mapsto \partial_v$  is a vector space isomorphism.

PROOF. Let  $v, w \in \mathbb{R}^m$  such that  $v \neq w$ . Choose an element  $u \in \mathbb{R}^m$  such that  $\langle u, v \rangle \neq \langle u, w \rangle$  and define  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  by  $f(x) = \langle u, x \rangle$ . Then  $\partial_v f = \langle u, v \rangle \neq \langle u, w \rangle = \partial_w f$  so  $\partial_v \neq \partial_w$ . This proves that the map  $\Phi$  is injective.

Let  $\alpha$  be an arbitrary element of  $T_p\mathbb{R}^m$ . For  $k = 1, \dots, m$  let  $\hat{x}_k : \mathbb{R}^m \rightarrow \mathbb{R}$  be the map given by

$$\hat{x}_k : (x_1, \dots, x_m) \mapsto x_k$$

and put  $v_k = \alpha(\hat{x}_k)$ . For the constant function  $1 : (x_1, \dots, x_m) \mapsto 1$  we have

$$\alpha(1) = \alpha(1 \cdot 1) = 1 \cdot \alpha(1) + 1 \cdot \alpha(1) = 2 \cdot \alpha(1),$$

so  $\alpha(1) = 0$ . By the linearity of  $\alpha$  it follows that  $\alpha(c) = 0$  for any constant  $c \in \mathbb{R}$ . Let  $f \in \varepsilon(p)$  and following Lemma 3.3 locally write

$$f(x) = f(p) + \sum_{k=1}^m (\hat{x}_k(x) - p_k) \cdot \psi_k(x),$$

where  $\psi_k \in \varepsilon(p)$  with

$$\psi_k(p) = \frac{\partial f}{\partial x_k}(p).$$

We can now apply the differential operator  $\alpha \in T_p\mathbb{R}^m$  and yield

$$\begin{aligned}
 \alpha(f) &= \alpha(f(p) + \sum_{k=1}^m (\hat{x}_k - p_k) \cdot \psi_k) \\
 &= \alpha(f(p)) + \sum_{k=1}^m \alpha(\hat{x}_k - p_k) \cdot \psi_k(p) + \sum_{k=1}^m (\hat{x}_k(p) - p_k) \cdot \alpha(\psi_k) \\
 &= \sum_{k=1}^m v_k \frac{\partial f}{\partial x_k}(p) \\
 &= \langle v, \text{grad} f_p \rangle \\
 &= \partial_v f,
 \end{aligned}$$

where  $v = (v_1, \dots, v_m) \in \mathbb{R}^m$ . This means that  $\Phi(v) = \partial_v = \alpha$  so the map  $\Phi : \mathbb{R}^m \rightarrow T_p\mathbb{R}^m$  is surjective and hence a vector space isomorphism.  $\square$

**Lemma 3.3.** *Let  $p$  be a point in  $\mathbb{R}^m$  and  $f : U \rightarrow \mathbb{R}$  be a differentiable function defined on an open ball around  $p$ . Then for each  $k = 1, 2, \dots, m$  there exist functions  $\psi_k : U \rightarrow \mathbb{R}$  such that*

$$f(x) = f(p) + \sum_{k=1}^m (x_k - p_k) \cdot \psi_k(x) \quad \text{and} \quad \psi_k(p) = \frac{\partial f}{\partial x_k}(p)$$

for all  $x \in U$ .

**PROOF.** It follows from the fundamental theorem of calculus that

$$\begin{aligned}
 f(x) - f(p) &= \int_0^1 \frac{\partial}{\partial t} (f(p + t(x - p))) dt \\
 &= \sum_{k=1}^m (x_k - p_k) \cdot \int_0^1 \frac{\partial f}{\partial x_k} (p + t(x - p)) dt.
 \end{aligned}$$

The statement then immediately follows by setting

$$\psi_k(x) = \int_0^1 \frac{\partial f}{\partial x_k} (p + t(x - p)) dt.$$

$\square$

**Remark 3.4.** Let  $p$  be a point in  $\mathbb{R}^m$ ,  $v \in T_p\mathbb{R}^m$  be a tangent vector at  $p$  and  $f : U \rightarrow \mathbb{R}$  be a  $C^1$ -function defined on an open subset  $U$  of  $\mathbb{R}^m$  containing  $p$ . Let  $\gamma : I \rightarrow U$  be a curve such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . The identification given by Theorem 3.2 tells us that  $v$  acts on  $f$  by

$$v(f) = \partial_v(f) = \langle v, \text{grad} f_p \rangle = df_p(\dot{\gamma}(0)) = \frac{d}{dt}(f \circ \gamma(t))|_{t=0}.$$

This implies that the real number  $v(f)$  is independent of the choice of the curve  $\gamma$  as long as  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ .

As a direct consequence of Theorem 3.2 we have the following useful result.

**Corollary 3.5.** *Let  $p$  be a point in  $\mathbb{R}^m$  and  $\{e_k \mid k = 1, \dots, m\}$  be a basis for  $\mathbb{R}^m$ . Then the set  $\{\partial_{e_k} \mid k = 1, \dots, m\}$  is a basis for the tangent space  $T_p\mathbb{R}^m$  at  $p$ .*

We shall now use the ideas presented above to generalize to the manifold setting. Let  $M$  be a differentiable manifold and for a point  $p \in M$  let  $\varepsilon(p)$  denote the set of differentiable real-valued functions defined on an open neighborhood of  $p$ .

**Definition 3.6.** Let  $M$  be a differentiable manifold and  $p$  be a point on  $M$ . A **tangent vector**  $X_p$  at  $p$  is a map  $X_p : \varepsilon(p) \rightarrow \mathbb{R}$  such that

- (i)  $X_p(\lambda \cdot f + \mu \cdot g) = \lambda \cdot X_p(f) + \mu \cdot X_p(g)$ ,
- (ii)  $X_p(f \cdot g) = X_p(f) \cdot g(p) + f(p) \cdot X_p(g)$

for all  $\lambda, \mu \in \mathbb{R}$  and  $f, g \in \varepsilon(p)$ . The set of tangent vectors at  $p$  is called the **tangent space** at  $p$  and denoted by  $T_pM$ .

The tangent space  $T_pM$  of  $M$  at  $p$  has the structure of a real vector space. The addition  $+$  and the multiplication  $\cdot$  by real numbers are simply given by

$$\begin{aligned} (X_p + Y_p)(f) &= X_p(f) + Y_p(f), \\ (\lambda \cdot X_p)(f) &= \lambda \cdot X_p(f) \end{aligned}$$

for all  $X_p, Y_p \in T_pM$ ,  $f \in \varepsilon(p)$  and  $\lambda \in \mathbb{R}$ .

**Remark 3.7.** Let  $M$  be an  $m$ -dimensional manifold and  $(U, x)$  be a local chart around  $p \in M$ . Then the differential  $dx_p : T_pM \rightarrow T_{x(p)}\mathbb{R}^m$  is a bijective linear map so for a given element  $X_p \in T_pM$  there exists a tangent vector  $v$  in  $\mathbb{R}^m$  such that  $dx_p(X_p) = v$ . The image  $x(U)$  is an open subset of  $\mathbb{R}^m$  containing  $x(p)$  so we can find a curve  $c : I \rightarrow x(U)$  with  $c(0) = x(p)$  and  $\dot{c}(0) = v$ . Then the composition  $\gamma = x^{-1} \circ c : I \rightarrow U$  is a curve in  $M$  through  $p$  since  $\gamma(0) = p$ . The element  $d(x^{-1})_{x(p)}(v)$  of the tangent space  $T_pM$  denoted by  $\dot{\gamma}(0)$  is called the **tangent** to the curve  $\gamma$  at  $p$ . It follows from the relation

$$\dot{\gamma}(0) = d(x^{-1})_{x(p)}(v) = X_p$$

that the tangent space  $T_pM$  can be thought of as the set of all tangents to curves through the point  $p$ .

If  $f : U \rightarrow \mathbb{R}$  is a  $C^1$ -function defined locally on  $U$  then it follows from Definition 3.14 that

$$\begin{aligned} X_p(f) &= (dx_p(X_p))(f \circ x^{-1}) \\ &= \frac{d}{dt}(f \circ x^{-1} \circ c(t))|_{t=0} \\ &= \frac{d}{dt}(f \circ \gamma(t))|_{t=0} \end{aligned}$$

It should be noted that the real number  $X_p(f)$  is independent of the choice of the chart  $(U, x)$  around  $p$  and the curve  $c : I \rightarrow x(U)$  as long as  $\gamma(0) = p$  and  $\dot{\gamma}(0) = X_p$ .

We shall now determine the tangent spaces of some of the explicit differentiable manifolds introduced in Chapter 2.

**Example 3.8.** Let  $\gamma : I \rightarrow S^m$  be a curve into the  $m$ -dimensional unit sphere in  $\mathbb{R}^{m+1}$  with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = X$ . The curve satisfies

$$\langle \gamma(t), \gamma(t) \rangle = 1$$

and differentiation yields

$$\langle \dot{\gamma}(t), \gamma(t) \rangle + \langle \gamma(t), \dot{\gamma}(t) \rangle = 0.$$

This means that  $\langle p, X \rangle = 0$  so every tangent vector  $X \in T_p S^m$  must be orthogonal to  $p$ . On the other hand if  $X \neq 0$  satisfies  $\langle p, X \rangle = 0$  then  $\gamma : \mathbb{R} \rightarrow S^m$  with

$$\gamma : t \mapsto \cos(t|X|) \cdot p + \sin(t|X|) \cdot X/|X|$$

is a curve into  $S^m$  with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = X$ . This shows that the tangent space  $T_p S^m$  is actually given by

$$T_p S^m = \{X \in \mathbb{R}^{m+1} \mid \langle p, X \rangle = 0\}.$$

**Proposition 3.9.** Let  $\text{Exp} : \mathbb{C}^{m \times m} \rightarrow \mathbb{C}^{m \times m}$  be the well-known exponential map for complex matrices given by the converging power series

$$\text{Exp} : Z \mapsto \sum_{k=0}^{\infty} \frac{Z^k}{k!}.$$

For two elements  $Z, W \in \mathbb{C}^{m \times m}$  we have the following

- (i)  $\det(\text{Exp}(Z)) = \exp(\text{trace} Z)$ ,
- (ii)  $\text{Exp}(Z^t) = \overline{\text{Exp}(Z)}^t$ ,
- (iii)  $\text{Exp}(\bar{Z}) = \overline{\text{Exp}(Z)}$ ,
- (iv) if  $ZW = WZ$  then  $\text{Exp}(Z + W) = \text{Exp}(Z)\text{Exp}(W)$ .

PROOF. See Exercise 3.2

□

The real general linear group  $\mathbf{GL}_m(\mathbb{R})$  is an open subset of  $\mathbb{R}^{m \times m}$  so its tangent space  $T_p\mathbf{GL}_m(\mathbb{R})$  at any point  $p$  is simply  $\mathbb{R}^{m \times m}$ . The tangent space  $T_e\mathbf{SL}_m(\mathbb{R})$  of the special linear group  $\mathbf{SL}_m(\mathbb{R})$  at the neutral element  $e$  can be determined as follows.

**Example 3.10.** If  $X$  is a matrix in  $\mathbb{R}^{m \times m}$  with  $\text{trace}X = 0$  then define a curve  $A : \mathbb{R} \rightarrow \mathbb{R}^{m \times m}$  by

$$A : s \mapsto \text{Exp}(sX).$$

Then  $A(0) = e$ ,  $\dot{A}(0) = X$  and

$$\det(A(s)) = \det(\text{Exp}(sX)) = \exp(\text{trace}(sX)) = \exp(0) = 1.$$

This shows that  $A$  is a curve into the special linear group  $\mathbf{SL}_m(\mathbb{R})$  and that  $X$  is an element of the tangent space  $T_e\mathbf{SL}_m(\mathbb{R})$  of  $\mathbf{SL}_m(\mathbb{R})$  at the neutral element  $e$ . Hence the linear space

$$\{X \in \mathbb{R}^{m \times m} \mid \text{trace}X = 0\}$$

of dimension  $m^2 - 1$  is contained in the tangent space  $T_e\mathbf{SL}_m(\mathbb{R})$ .

The curve given by  $s \mapsto \text{Exp}(se) = \exp(s)e$  is not contained in  $\mathbf{SL}_m(\mathbb{R})$  so the dimension of  $T_e\mathbf{SL}_m(\mathbb{R})$  is at most  $m^2 - 1$ . This shows that

$$T_e\mathbf{SL}_m(\mathbb{R}) = \{X \in \mathbb{R}^{m \times m} \mid \text{trace}X = 0\}.$$

**Example 3.11.** Let  $\gamma : I \rightarrow \mathbf{O}(m)$  be a curve into the orthogonal group  $\mathbf{O}(m)$  such that  $\gamma(0) = e$ . Then  $\gamma(s)^t\gamma(s) = e$  for all  $s \in I$  and differentiation gives

$$\{\dot{\gamma}(s)^t\gamma(s) + \gamma(s)^t\dot{\gamma}(s)\}|_{s=0} = 0$$

or equivalently  $\dot{\gamma}(0)^t + \dot{\gamma}(0) = 0$ . This means that each tangent vector of  $\mathbf{O}(m)$  at  $e$  is a skew-symmetric matrix.

On the other hand, for an arbitrary real skew-symmetric matrix  $X$  define the curve  $A : \mathbb{R} \rightarrow \mathbb{R}^{m \times m}$  by  $A : s \mapsto \text{Exp}(sX)$ . Then

$$\begin{aligned} A(s)^tA(s) &= \text{Exp}(sX)^t\text{Exp}(sX) \\ &= \text{Exp}(sX^t)\text{Exp}(sX) \\ &= \text{Exp}(s(X^t + X)) \\ &= \text{Exp}(0) \\ &= e. \end{aligned}$$

This shows that  $A$  is a curve on the orthogonal group,  $A(0) = e$  and  $\dot{A}(0) = X$  so  $X$  is an element of  $T_e\mathbf{O}(m)$ . Hence

$$T_e\mathbf{O}(m) = \{X \in \mathbb{R}^{m \times m} \mid X^t + X = 0\}.$$

The dimension of  $T_e\mathbf{O}(m)$  is therefore  $m(m-1)/2$ . The orthogonal group  $\mathbf{O}(m)$  is diffeomorphic to  $\mathbf{SO}(m) \times \{\pm 1\}$  so  $\dim(\mathbf{SO}(m)) = \dim(\mathbf{O}(m))$  hence

$$T_e\mathbf{SO}(m) = T_e\mathbf{O}(m) = \{X \in \mathbb{R}^{m \times m} \mid X^t + X = 0\}.$$

We have proven the following result.

**Proposition 3.12.** *Let  $e$  be the neutral element of the classical real Lie groups  $\mathbf{GL}_m(\mathbb{R})$ ,  $\mathbf{SL}_m(\mathbb{R})$ ,  $\mathbf{O}(m)$ ,  $\mathbf{SO}(m)$ . Then their tangent spaces at  $e$  are given by*

$$\begin{aligned} T_e\mathbf{GL}_m(\mathbb{R}) &= \mathbb{R}^{m \times m} \\ T_e\mathbf{SL}_m(\mathbb{R}) &= \{X \in \mathbb{R}^{m \times m} \mid \text{trace}X = 0\} \\ T_e\mathbf{O}(m) &= \{X \in \mathbb{R}^{m \times m} \mid X^t + X = 0\} \\ T_e\mathbf{SO}(m) &= T_e\mathbf{O}(m) \cap T_e\mathbf{SL}_m(\mathbb{R}) = T_e\mathbf{O}(m) \end{aligned}$$

For the classical complex Lie groups similar methods can be used to prove the following.

**Proposition 3.13.** *Let  $e$  be the neutral element of the classical complex Lie groups  $\mathbf{GL}_m(\mathbb{C})$ ,  $\mathbf{SL}_m(\mathbb{C})$ ,  $\mathbf{U}(m)$ , and  $\mathbf{SU}(m)$ . Then their tangent spaces at  $e$  are given by*

$$\begin{aligned} T_e\mathbf{GL}_m(\mathbb{C}) &= \mathbb{C}^{m \times m} \\ T_e\mathbf{SL}_m(\mathbb{C}) &= \{Z \in \mathbb{C}^{m \times m} \mid \text{trace}Z = 0\} \\ T_e\mathbf{U}(m) &= \{Z \in \mathbb{C}^{m \times m} \mid \bar{Z}^t + Z = 0\} \\ T_e\mathbf{SU}(m) &= T_e\mathbf{U}(m) \cap T_e\mathbf{SL}_m(\mathbb{C}). \end{aligned}$$

PROOF. See Exercise 3.4 □

**Definition 3.14.** Let  $\phi : M \rightarrow N$  be a differentiable map between manifolds. Then the **differential**  $d\phi_p$  of  $\phi$  at a point  $p$  in  $M$  is the map  $d\phi_p : T_pM \rightarrow T_{\phi(p)}N$  such that for all  $X_p \in T_pM$  and  $f \in \varepsilon(\phi(p))$

$$(d\phi_p(X_p))(f) = X_p(f \circ \phi).$$

**Remark 3.15.** Let  $M$  and  $N$  be differentiable manifolds,  $p \in M$  and  $\phi : M \rightarrow N$  be a smooth map. Further let  $\gamma : I \rightarrow M$  be a curve on  $M$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = X_p$ . Let  $c : I \rightarrow N$  be the curve  $c = \phi \circ \gamma$  in  $N$  with  $c(0) = \phi(p)$  and put  $Y_{\phi(p)} = \dot{c}(0)$ . Then it is an immediate consequence of Definition 3.14 that for each function  $f \in \varepsilon(\phi(p))$  defined locally around  $\phi(p)$

$$(d\phi_p(X_p))(f) = Y_{\phi(p)}(f)$$

$$\begin{aligned}
&= \frac{d}{dt}(f \circ \phi \circ \gamma(t))|_{t=0} \\
&= \frac{d}{dt}(f \circ c(t))|_{t=0} \\
&= Y_{\phi(p)}(f).
\end{aligned}$$

Hence  $d\phi_p(X_p) = Y_{\phi(p)}$  or equivalently  $d\phi_p(\dot{\gamma}(0)) = \dot{c}(0)$ . This result should be compared with Remark 2.10.

**Proposition 3.16.** *Let  $\phi : M_1 \rightarrow M_2$  and  $\psi : M_2 \rightarrow M_3$  be differentiable maps between manifolds, then for each  $p \in M_1$  we have*

- (i) *the map  $d\phi_p : T_p M_1 \rightarrow T_{\phi(p)} M_2$  is linear,*
- (ii) *if  $\text{id}_{M_1} : M_1 \rightarrow M_1$  is the identity map, then  $d(\text{id}_{M_1})_p = \text{id}_{T_p M_1}$ ,*
- (iii)  *$d(\psi \circ \phi)_p = d\psi_{\phi(p)} \circ d\phi_p$ .*

PROOF. The only non-trivial statement is the relation (iii) which is called the **chain rule**. If  $X_p \in T_p M_1$  and  $f \in \varepsilon(\psi \circ \phi(p))$ , then

$$\begin{aligned}
(d\psi_{\phi(p)} \circ d\phi_p)(X_p)(f) &= (d\psi_{\phi(p)}(d\phi_p(X_p)))(f) \\
&= (d\phi_p(X_p))(f \circ \psi) \\
&= X_p(f \circ \psi \circ \phi) \\
&= (d(\psi \circ \phi)_p(X_p))(f).
\end{aligned}$$

□

**Corollary 3.17.** *Let  $\phi : M \rightarrow N$  be a diffeomorphism with inverse  $\psi = \phi^{-1} : N \rightarrow M$ . Then the differential  $d\phi_p : T_p M \rightarrow T_{\phi(p)} N$  at  $p$  is bijective and  $(d\phi_p)^{-1} = d\psi_{\phi(p)}$ .*

PROOF. The statement is a direct consequence of the following relations

$$\begin{aligned}
d\psi_{\phi(p)} \circ d\phi_p &= d(\psi \circ \phi)_p = d(\text{id}_M)_p = \text{id}_{T_p M}, \\
d\phi_p \circ d\psi_{\phi(p)} &= d(\phi \circ \psi)_{\phi(p)} = d(\text{id}_N)_{\phi(p)} = \text{id}_{T_{\phi(p)} N}.
\end{aligned}$$

□

We are now ready to prove the following interesting result. This is of course a direct generalization of the corresponding result in the classical theory for surfaces in  $\mathbb{R}^3$ .

**Theorem 3.18.** *Let  $M^m$  be an  $m$ -dimensional differentiable manifold and  $p$  be a point in  $M$ . Then the tangent space  $T_p M$  at  $p$  is an  $m$ -dimensional real vector space.*

PROOF. Let  $(U, x)$  be a local chart on  $M$ . Then the linear map  $dx_p : T_p M \rightarrow T_{x(p)} \mathbb{R}^m$  is a vector space isomorphism. The statement now follows from Theorem 3.2 and Corollary 3.17. □



**Proposition 3.19.** *Let  $M^m$  be a differentiable manifold,  $(U, x)$  be a local chart on  $M$  and  $\{e_k \mid k = 1, \dots, m\}$  be the canonical basis for  $\mathbb{R}^m$ . For an arbitrary point  $p$  in  $U$  we define  $(\frac{\partial}{\partial x_k})_p$  in  $T_pM$  by*

$$\left(\frac{\partial}{\partial x_k}\right)_p : f \mapsto \frac{\partial f}{\partial x_k}(p) = \partial_{e_k}(f \circ x^{-1})(x(p)).$$

Then the set

$$\left\{\left(\frac{\partial}{\partial x_k}\right)_p \mid k = 1, 2, \dots, m\right\}$$

is a basis for the tangent space  $T_pM$  of  $M$  at  $p$ .

**PROOF.** The local chart  $x : U \rightarrow x(U)$  is a diffeomorphism and the differential  $(dx^{-1})_{x(p)} : T_{x(p)}\mathbb{R}^m \rightarrow T_pM$  of the inverse  $x^{-1} : x(U) \rightarrow U$  satisfies

$$\begin{aligned} (dx^{-1})_{x(p)}(\partial_{e_k})(f) &= \partial_{e_k}(f \circ x^{-1})(x(p)) \\ &= \left(\frac{\partial}{\partial x_k}\right)_p(f) \end{aligned}$$

for all  $f \in \varepsilon(p)$ . The statement is then a direct consequence of Corollary 3.5.  $\square$

The rest of this chapter is devoted to the introduction of special types of differentiable maps, the immersions, the submersions and the embeddings.

**Definition 3.20.** A differentiable map  $\phi : M \rightarrow N$  between manifolds is said to be an **immersion** if for each  $p \in M$  the differential  $d\phi_p : T_pM \rightarrow T_{\phi(p)}N$  is injective. An **embedding** is an immersion  $\phi : M \rightarrow N$  which is a homeomorphism onto its image  $\phi(M)$ .

For positive integers  $m, n$  with  $m < n$  we have the inclusion map  $\phi : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{n+1}$  given by

$$\phi : (x_1, \dots, x_{m+1}) \mapsto (x_1, \dots, x_{m+1}, 0, \dots, 0).$$

The differential  $d\phi_x$  at  $x$  is injective since  $d\phi_x(v) = (v, 0)$ . The map  $\phi$  is obviously a homeomorphism onto its image  $\phi(\mathbb{R}^{m+1})$  hence an embedding. It is easily seen that even the restriction  $\phi|_{S^m} : S^m \rightarrow S^n$  of  $\phi$  to the  $m$ -dimensional unit sphere  $S^m$  in  $\mathbb{R}^{m+1}$  is an embedding.

**Definition 3.21.** Let  $M$  be an  $m$ -dimensional differentiable manifold and  $U$  be an open subset of  $\mathbb{R}^m$ . An immersion  $\phi : U \rightarrow M$  is called a local **parametrization** of  $M$ .

If  $M$  is a differentiable manifold and  $(U, x)$  is a chart on  $M$ , then the inverse  $x^{-1} : x(U) \rightarrow U$  of  $x$  is a parametrization of the subset  $U$  of  $M$ .

**Example 3.22.** Let  $S^1$  be the unit circle in the complex plane  $\mathbb{C}$ . For a non-zero integer  $k \in \mathbb{Z}$  define  $\phi_k : S^1 \rightarrow \mathbb{C}$  by  $\phi_k : z \mapsto z^k$ . For a point  $w \in S^1$  let  $\gamma_w : \mathbb{R} \rightarrow S^1$  be the curve with  $\gamma_w : t \mapsto we^{it}$ . Then  $\gamma_w(0) = w$  and  $\dot{\gamma}_w(0) = iw$ . For the differential of  $\phi_k$  we have

$$(d\phi_k)_w(\dot{\gamma}_w(0)) = \frac{d}{dt}(\phi_k \circ \gamma_w(t))|_{t=0} = \frac{d}{dt}(w^k e^{ikt})|_{t=0} = kiw^k \neq 0.$$

This shows that the differential  $(d\phi_k)_w : T_w S^1 \cong \mathbb{R} \rightarrow T_w \mathbb{C} \cong \mathbb{R}^2$  is injective, so the map  $\phi_k$  is an immersion. It is easily seen that  $\phi_k$  is an embedding if and only if  $k = \pm 1$ .

**Example 3.23.** Let  $q \in S^3$  be a quaternion of unit length and  $\phi_q : S^1 \rightarrow S^3$  be the map defined by  $\phi_q : z \mapsto qz$ . For  $w \in S^1$  let  $\gamma_w : \mathbb{R} \rightarrow S^1$  be the curve given by  $\gamma_w(t) = we^{it}$ . Then  $\gamma_w(0) = w$ ,  $\dot{\gamma}_w(0) = iw$  and  $\phi_q(\gamma_w(t)) = qwe^{it}$ . By differentiating we yield

$$d\phi_q(\dot{\gamma}_w(0)) = \frac{d}{dt}(\phi_q(\gamma_w(t)))|_{t=0} = \frac{d}{dt}(qwe^{it})|_{t=0} = qiw.$$

Then  $|d\phi_q(\dot{\gamma}_w(0))| = |qwi| = |q||w| = 1 \neq 0$  implies that the differential  $d\phi_q$  is injective. It is easily checked that the immersion  $\phi_q$  is an embedding.

In the next example we construct an interesting embedding of the real projective space  $\mathbb{R}P^m$  into the vector space  $\text{Sym}(\mathbb{R}^{m+1})$  of the real symmetric  $(m+1) \times (m+1)$  matrices.

**Example 3.24.** Let  $S^m$  be the  $m$ -dimensional unit sphere in  $\mathbb{R}^{m+1}$ . For a point  $p \in S^m$  let

$$l_p = \{\lambda p \in \mathbb{R}^{m+1} \mid \lambda \in \mathbb{R}\}$$

be the line through the origin generated by  $p$  and  $\rho_p : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1}$  be the reflection about the line  $l_p$ . Then  $\rho_p$  is an element of  $\text{End}(\mathbb{R}^{m+1})$  i.e. the set of linear endomorphisms of  $\mathbb{R}^{m+1}$  which can be identified with  $\mathbb{R}^{(m+1) \times (m+1)}$ . It is easily checked that the reflection about the line  $l_p$  is given by

$$\rho_p : q \mapsto 2\langle q, p \rangle p - q.$$

It then follows from the equations

$$\rho_p(q) = 2\langle q, p \rangle p - q = 2p\langle p, q \rangle - q = (2pp^t - e)q$$

that the matrix in  $\mathbb{R}^{(m+1) \times (m+1)}$  corresponding to  $\rho_p$  is just the symmetric

$$(2pp^t - e).$$

We will now show that the map  $\phi : S^m \rightarrow \text{Sym}(\mathbb{R}^{m+1})$  given by

$$\phi : p \mapsto \rho_p$$

is an immersion. Let  $p$  be an arbitrary point on  $S^m$  and  $\alpha, \beta : I \rightarrow S^m$  be two curves meeting at  $p$ , that is  $\alpha(0) = p = \beta(0)$ , with  $X = \dot{\alpha}(0)$  and  $Y = \dot{\beta}(0)$ . For  $\gamma \in \{\alpha, \beta\}$  we have

$$\phi \circ \gamma : t \mapsto (q \mapsto 2\langle q, \gamma(t) \rangle \gamma(t) - q)$$

so

$$\begin{aligned} (d\phi)_p(\dot{\gamma}(0)) &= \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} \\ &= (q \mapsto 2\langle q, \dot{\gamma}(0) \rangle \gamma(0) + 2\langle q, \gamma(0) \rangle \dot{\gamma}(0)). \end{aligned}$$

This means that

$$d\phi_p(X) = (q \mapsto 2\langle q, X \rangle p + 2\langle q, p \rangle X)$$

and

$$d\phi_p(Y) = (q \mapsto 2\langle q, Y \rangle p + 2\langle q, p \rangle Y).$$

If  $X \neq Y$  then  $d\phi_p(X) \neq d\phi_p(Y)$  so the differential  $d\phi_p$  is injective, hence the map  $\phi : S^m \rightarrow \text{Sym}(\mathbb{R}^{m+1})$  is an immersion.

If two points  $p, q \in S^m$  are linearly independent, then the lines  $l_p$  and  $l_q$  are different. But these are just the eigenspaces of  $\rho_p$  and  $\rho_q$  with the eigenvalue  $+1$ , respectively. This shows that the linear endomorphisms  $\rho_p, \rho_q$  of  $\mathbb{R}^{m+1}$  are different in this case.

On the other hand, if  $p$  and  $q$  are parallel then  $p = \pm q$  hence  $\rho_p = \rho_q$ . This means that the image  $\phi(S^m)$  can be identified with the quotient space  $S^m / \equiv$  where  $\equiv$  is the equivalence relation defined by

$$x \equiv y \text{ if and only if } x = \pm y.$$

This is of course the real projective space  $\mathbb{R}P^m$  so the map  $\phi$  induces an embedding  $\Phi : \mathbb{R}P^m \rightarrow \text{Sym}(\mathbb{R}^{m+1})$  with

$$\Phi : [p] \rightarrow \rho_p.$$

For each  $p \in S^m$  the reflection  $\rho_p : \mathbb{R}^{m+1} \rightarrow \mathbb{R}^{m+1}$  about the line  $l_p$  satisfies

$$\rho_p^t \cdot \rho_p = e.$$

This shows that the image  $\Phi(\mathbb{R}P^m) = \phi(S^m)$  is not only contained in the linear space  $\text{Sym}(\mathbb{R}^{m+1})$  but also in the orthogonal group  $\mathbf{O}(m+1)$  which we know is a submanifold of  $\mathbb{R}^{(m+1) \times (m+1)}$ .

The following result was proven by Hassler Whitney in his famous paper, *Differentiable Manifolds*, Ann. of Math. **37** (1936), 645-680.

**Deep Result 3.25.** *For  $1 \leq r \leq \infty$  let  $M$  be an  $m$ -dimensional  $C^r$ -manifold. Then there exists a  $C^r$ -embedding  $\phi : M \rightarrow \mathbb{R}^{2m+1}$  of  $M$  into the  $(2m+1)$ -dimensional real vector space  $\mathbb{R}^{2m+1}$ .*

The classical inverse function theorem generalizes to the manifold setting as follows.

**Theorem 3.26** (The Inverse Function Theorem). *Let  $\phi : M \rightarrow N$  be a differentiable map between manifolds with  $\dim M = \dim N$ . If  $p$  is a point in  $M$  such that the differential  $d\phi_p : T_p M \rightarrow T_{\phi(p)} N$  at  $p$  is bijective then there exist open neighborhoods  $U_p$  around  $p$  and  $U_q$  around  $q = \phi(p)$  such that  $\psi = \phi|_{U_p} : U_p \rightarrow U_q$  is bijective and the inverse  $\psi^{-1} : U_q \rightarrow U_p$  is differentiable.*

PROOF. See Exercise 3.8 □

We shall now generalize the classical implicit function theorem to manifolds. For this we need the following definition.

**Definition 3.27.** Let  $m, n$  be positive integers and  $\phi : M^m \rightarrow N^n$  be a differentiable map between manifolds. A point  $p \in M$  is said to be **critical** for  $\phi$  if the differential

$$d\phi_p : T_p M \rightarrow T_{\phi(p)} N$$

is not of full rank, and **regular** if it is not critical. A point  $q \in \phi(M)$  is called a **regular value** for  $\phi$  if every point of the pre-image  $\phi^{-1}(\{q\})$  of  $\{q\}$  is regular and a **critical value** otherwise.

**Theorem 3.28** (The Implicit Function Theorem). *Let  $\phi : M^m \rightarrow N^n$  be a differentiable map between manifolds such that  $m > n$ . If  $q \in \phi(M)$  is a regular value, then the pre-image  $\phi^{-1}(\{q\})$  of  $q$  is an  $(m-n)$ -dimensional submanifold of  $M^m$ . The tangent space  $T_p \phi^{-1}(\{q\})$  of  $\phi^{-1}(\{q\})$  at  $p$  is the kernel of the differential  $d\phi_p$  i.e.*

$$T_p \phi^{-1}(\{q\}) = \{X \in T_p M \mid d\phi_p(X) = 0\}.$$

PROOF. Let  $(V, y)$  be a chart on  $N$  with  $q \in V$  and  $y(q) = 0$ . For a point  $p \in \phi^{-1}(\{q\})$  we choose a chart  $(U, x)$  on  $M$  such that  $p \in U$ ,  $x(p) = 0$  and  $\phi(U) \subset V$ . Then the differential of the map

$$\psi = y \circ \phi \circ x^{-1}|_{x(U)} : x(U) \rightarrow \mathbb{R}^n$$

at the point 0 is given by

$$d\psi_0 = (dy)_q \circ d\phi_p \circ (dx^{-1})_0 : T_0 \mathbb{R}^m \rightarrow T_0 \mathbb{R}^n.$$

The pairs  $(U, x)$  and  $(V, y)$  are charts so the differentials  $(dy)_q$  and  $(dx^{-1})_0$  are bijective. This means that  $d\psi_0$  is surjective since  $d\phi_p$  is. It then follows from Theorem 2.14 that  $x(\phi^{-1}(\{q\}) \cap U)$  is an  $(m-n)$ -dimensional submanifold of  $x(U)$ . Hence  $\phi^{-1}(\{q\}) \cap U$  is an  $(m-n)$ -dimensional submanifold of  $U$ . This is true for each point  $p \in \phi^{-1}(\{q\})$

so we have proven that  $\phi^{-1}(\{q\})$  is an  $(m-n)$ -dimensional submanifold of  $M^m$ .

Let  $\gamma : I \rightarrow \phi^{-1}(\{q\})$  be a curve such that  $\gamma(0) = p$ . Then

$$(d\phi)_p(\dot{\gamma}(0)) = \frac{d}{dt}(\phi \circ \gamma(t))|_{t=0} = \frac{dq}{dt}|_{t=0} = 0.$$

This implies that  $T_p\phi^{-1}(\{q\})$  is contained in and has the same dimension as the kernel of  $d\phi_p$ , so  $T_p\phi^{-1}(\{q\}) = \text{Ker } d\phi_p$ .  $\square$

**Definition 3.29.** For positive integers  $m, n$  with  $m \geq n$  a map  $\phi : M^m \rightarrow N^n$  between two manifolds is said to be a **submersion** if for each  $p \in M$  the differential  $d\phi_p : T_pM \rightarrow T_{\phi(p)}N$  is surjective.

If  $m, n \in \mathbb{N}$  such that  $m \geq n$  then we have the projection map  $\pi : \mathbb{R}^m \rightarrow \mathbb{R}^n$  given by  $\pi : (x_1, \dots, x_m) \mapsto (x_1, \dots, x_n)$ . Its differential  $d\pi_x$  at a point  $x$  is surjective since

$$d\pi_x(v_1, \dots, v_m) = (v_1, \dots, v_n).$$

This means that the projection is a submersion. An important submersion between spheres is given by the following.

**Example 3.30.** Let  $S^3$  and  $S^2$  be the unit spheres in  $\mathbb{C}^2$  and  $\mathbb{C} \times \mathbb{R} \cong \mathbb{R}^3$ , respectively. The **Hopf map**  $\phi : S^3 \rightarrow S^2$  is given by

$$\phi : (x, y) \mapsto (2x\bar{y}, |x|^2 - |y|^2).$$

For  $p \in S^3$  the **Hopf circle**  $C_p$  through  $p$  is given by

$$C_p = \{e^{i\theta}(x, y) \mid \theta \in \mathbb{R}\}.$$

The following shows that the Hopf map is constant along each Hopf circle

$$\begin{aligned} \phi(e^{i\theta}(x, y)) &= (2e^{i\theta}xe^{-i\theta}\bar{y}, |e^{i\theta}x|^2 - |e^{i\theta}y|^2) \\ &= (2x\bar{y}, |x|^2 - |y|^2) \\ &= \phi((x, y)). \end{aligned}$$

The map  $\phi$  and its differential  $d\phi_p : T_pS^3 \rightarrow T_{\phi(p)}S^2$  are surjective for each  $p \in S^3$ . This implies that each point  $q \in S^2$  is a regular value and the fibres of  $\phi$  are 1-dimensional submanifolds of  $S^3$ . They are actually the great circles given by

$$\phi^{-1}(\{(2x\bar{y}, |x|^2 - |y|^2)\}) = \{e^{i\theta}(x, y) \mid \theta \in \mathbb{R}\}.$$

This means that the 3-dimensional sphere  $S^3$  is a disjoint union of great circles

$$S^3 = \bigcup_{q \in S^2} \phi^{-1}(\{q\}).$$

## Exercises

**Exercise 3.1.** Let  $p$  be an arbitrary point on the unit sphere  $S^{2n+1}$  of  $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ . Determine the tangent space  $T_p S^{2n+1}$  and show that it contains an  $n$ -dimensional complex subspace of  $\mathbb{C}^{n+1}$ .

**Exercise 3.2.** Use your local library to find a proof of Proposition 3.9.

**Exercise 3.3.** Prove that the matrices

$$X_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

form a basis for the tangent space  $T_e \mathbf{SL}_2(\mathbb{R})$  of the real special linear group  $\mathbf{SL}_2(\mathbb{R})$  at the neutral element  $e$ . For each  $k = 1, 2, 3$  find an explicit formula for the curve  $\gamma_k : \mathbb{R} \rightarrow \mathbf{SL}_2(\mathbb{R})$  given by

$$\gamma_k : s \mapsto \text{Exp}(sX_k).$$

**Exercise 3.4.** Find a proof of Proposition 3.13.

**Exercise 3.5.** Prove that the matrices

$$Z_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Z_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Z_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

form a basis for the tangent space  $T_e \mathbf{SU}(2)$  of the special unitary group  $\mathbf{SU}(2)$  at the neutral element  $e$ . For each  $k = 1, 2, 3$  find an explicit formula for the curve  $\gamma_k : \mathbb{R} \rightarrow \mathbf{SU}(2)$  given by

$$\gamma_k : s \mapsto \text{Exp}(sZ_k).$$

**Exercise 3.6.** For each  $k \in \mathbb{N}_0$  define  $\phi_k : \mathbb{C} \rightarrow \mathbb{C}$  and  $\psi_k : \mathbb{C}^* \rightarrow \mathbb{C}$  by  $\phi_k, \psi_k : z \mapsto z^k$ . For which  $k \in \mathbb{N}_0$  are  $\phi_k, \psi_k$  immersions, submersions or embeddings.

**Exercise 3.7.** Prove that the map  $\phi : \mathbb{R}^m \rightarrow \mathbb{C}^m$  given by

$$\phi : (x_1, \dots, x_m) \mapsto (e^{ix_1}, \dots, e^{ix_m})$$

is a parametrization of the  $m$ -dimensional torus  $T^m$  in  $\mathbb{C}^m$ .

**Exercise 3.8.** Find a proof of Theorem 3.26.

**Exercise 3.9.** Prove that the Hopf-map  $\phi : S^3 \rightarrow S^2$  with  $\phi : (x, y) \mapsto (2x\bar{y}, |x|^2 - |y|^2)$  is a submersion.

## The Tangent Bundle

In this chapter we introduce the tangent bundle  $TM$  of a differentiable manifold  $M$ . Intuitively, this is the object that we get by glueing at each point  $p$  of  $M$  the corresponding tangent space  $T_pM$ . The differentiable structure on  $M$  induces a natural differentiable structure on the tangent bundle  $TM$  turning it into a differentiable manifold.

We have already seen that for a point  $p \in \mathbb{R}^m$  the **tangent space**  $T_p\mathbb{R}^m$  can be identified with the  $m$ -dimensional vector space  $\mathbb{R}^m$ . This means that if we at each point  $p \in \mathbb{R}^m$  glue the tangent space  $T_p\mathbb{R}^m$  to  $\mathbb{R}^m$  we obtain the so called **tangent bundle** of  $\mathbb{R}^m$

$$T\mathbb{R}^m = \{(p, v) \mid p \in \mathbb{R}^m \text{ and } v \in T_p\mathbb{R}^m\}.$$

For this we have the **natural projection**  $\pi : T\mathbb{R}^m \rightarrow \mathbb{R}^m$  defined by

$$\pi : (p, v) \mapsto p$$

and for each point  $p \in M$  the fibre  $\pi^{-1}(\{p\})$  over  $p$  is precisely the tangent space  $T_p\mathbb{R}^m$  at  $p$ .

Classically, a **vector field**  $X$  on  $\mathbb{R}^m$  is a smooth map  $X : \mathbb{R}^m \rightarrow \mathbb{R}^m$  but we would like to view it as a map  $X : \mathbb{R}^m \rightarrow T\mathbb{R}^m$  into the tangent bundle and write

$$X : p \mapsto (p, X_p).$$

Following Proposition 3.19 two vector fields  $X, Y : \mathbb{R}^m \rightarrow T\mathbb{R}^m$  can be written as

$$X = \sum_{k=1}^m a_k \frac{\partial}{\partial x_k} \quad \text{and} \quad Y = \sum_{k=1}^m b_k \frac{\partial}{\partial x_k},$$

where  $a_k, b_k : \mathbb{R}^m \rightarrow \mathbb{R}$  are smooth functions defined on  $\mathbb{R}^m$ . If  $f : \mathbb{R}^m \rightarrow \mathbb{R}$  is another such function the **commutator**  $[X, Y]$  acts on  $f$  as follows:

$$\begin{aligned} [X, Y](f) &= X(Y(f)) - Y(X(f)) \\ &= \sum_{k,l=1}^m \left( a_k \frac{\partial}{\partial x_k} \left( b_l \frac{\partial}{\partial x_l} \right) - b_k \frac{\partial}{\partial x_k} \left( a_l \frac{\partial}{\partial x_l} \right) \right) (f) \end{aligned}$$

$$\begin{aligned}
&= \sum_{k,l=1}^m \left( a_k \frac{\partial b_l}{\partial x_k} \frac{\partial}{\partial x_l} + a_k b_l \frac{\partial^2}{\partial x_k \partial x_l} \right. \\
&\quad \left. - b_k \frac{\partial a_l}{\partial x_k} \frac{\partial}{\partial x_l} - b_k a_l \frac{\partial^2}{\partial x_k \partial x_l} \right) (f) \\
&= \sum_{l=1}^m \left[ \sum_{k=1}^m \left( a_k \frac{\partial b_l}{\partial x_k} - b_k \frac{\partial a_l}{\partial x_k} \right) \right] \frac{\partial}{\partial x_l} (f).
\end{aligned}$$

This shows that the commutator  $[X, Y]$  is a smooth vector field on  $\mathbb{R}^m$ .

We shall now generalize to the manifold setting. First we introduce the following notion of a topological vector bundle.

**Definition 4.1.** Let  $E$  and  $M$  be topological manifolds and  $\pi : E \rightarrow M$  be a continuous surjective map. The triple  $(E, M, \pi)$  is called an  $n$ -dimensional **topological vector bundle** over  $M$  if (i) for each  $p \in M$  the fibre  $E_p = \pi^{-1}(\{p\})$  is an  $n$ -dimensional vector space, (ii) for each  $p \in M$  there exists a **bundle chart**  $(\pi^{-1}(U), \psi)$  consisting of the pre-image  $\pi^{-1}(U)$  of an open neighbourhood  $U$  of  $p$  and a homeomorphism  $\psi : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$  such that for all  $q \in U$  the map  $\psi_q = \psi|_{E_q} : E_q \rightarrow \{q\} \times \mathbb{R}^n$  is a vector space isomorphism. A **bundle atlas** for  $(E, M, \pi)$  is a collection

$$\mathcal{B} = \{(\pi^{-1}(U_\alpha), \psi_\alpha) \mid \alpha \in I\}$$

of bundle charts such that  $M = \cup_\alpha U_\alpha$ . For  $\alpha, \beta \in I$  there exists a map  $A_{\alpha,\beta} : U_\alpha \cap U_\beta \rightarrow \mathbf{GL}_n(\mathbb{R})$  such that the corresponding continuous map

$$\psi_\beta \circ \psi_\alpha^{-1}|_{(U_\alpha \cap U_\beta) \times \mathbb{R}^n} : (U_\alpha \cap U_\beta) \times \mathbb{R}^n \rightarrow (U_\alpha \cap U_\beta) \times \mathbb{R}^n$$

is given by

$$(p, v) \mapsto (p, (A_{\alpha,\beta}(p))(v)).$$

The elements of  $\{A_{\alpha,\beta} \mid \alpha, \beta \in I\}$  are called the **transition maps** of the bundle atlas  $\mathcal{B}$ .

**Definition 4.2.** Let  $(E, M, \pi)$  be an  $n$ -dimensional topological vector bundle over  $M$ . A continuous map  $\sigma : M \rightarrow E$  is called a **section** of the bundle  $(E, M, \pi)$  if  $\pi \circ \sigma(p) = p$  for each  $p \in M$ .

**Definition 4.3.** A topological vector bundle  $(E, M, \pi)$  over  $M$ , of dimension  $n$ , is said to be **trivial** if there exists a global bundle chart  $\psi : E \rightarrow M \times \mathbb{R}^n$ .

**Example 4.4.** For  $n \in \mathbb{Z}^+$  and a topological manifold  $M$  we have the  $n$ -dimensional **trivial vector bundle**  $(M \times \mathbb{R}^n, M, \pi)$  over  $M$ , where  $\pi : M \times \mathbb{R}^n \rightarrow M$  is the projection map with  $\pi : (p, v) \mapsto p$ . The



bundle is trivial since the identity map  $\psi : M \times \mathbb{R}^n \rightarrow M \times \mathbb{R}^n$  is a global bundle chart.

**Definition 4.5.** Let  $E$  and  $M$  be **differentiable** manifolds and  $\pi : E \rightarrow M$  be a **differentiable** map such that  $(E, M, \pi)$  is an  $n$ -dimensional topological vector bundle. A bundle atlas  $\mathcal{B}$  for  $(E, M, \pi)$  is said to be differentiable if the corresponding transition maps are differentiable. A **differentiable vector bundle** is a topological vector bundle together with a maximal differentiable bundle atlas. By  $C^\infty(E)$  we denote the set of all smooth sections of  $(E, M, \pi)$ .

**From now on we shall assume, when not stating otherwise, that all our vector bundles are smooth.**

**Definition 4.6.** Let  $(E, M, \pi)$  be a vector bundle over a manifold  $M$ . Then we define the operations  $+$  and  $\cdot$  on the set  $C^\infty(E)$  of smooth sections of  $(E, M, \pi)$  by

- (i)  $(v + w)_p = v_p + w_p$ ,
- (ii)  $(f \cdot v)_p = f(p) \cdot v_p$

for all  $v, w \in C^\infty(E)$  and  $f \in C^\infty(M)$ . If  $U$  is an open subset of  $M$  then a set  $\{v_1, \dots, v_n\}$  of smooth sections  $v_1, \dots, v_n : U \rightarrow E$  on  $U$  is said to be a **local frame** for  $E$  if for each  $p \in U$  the set  $\{(v_1)_p, \dots, (v_n)_p\}$  is a basis for the vector space  $E_p$ .

According to Definition 2.18, the set of smooth real-valued functions on  $M$  is denoted by  $C^\infty(M)$ . With the above defined operations on  $C^\infty(E)$  it becomes a module over  $C^\infty(M)$  and in particular a vector space over the real numbers as the constant functions in  $C^\infty(M)$ .

**Example 4.7.** Let  $M^m$  be a differentiable manifold with maximal atlas  $\hat{\mathcal{A}}$ . Define the set  $TM$  by

$$TM = \{(p, v) \mid p \in M \text{ and } v \in T_p M\}$$

and let  $\pi : TM \rightarrow M$  be the **projection map** satisfying

$$\pi : (p, v) \mapsto p.$$

Then the fibre  $\pi^{-1}(\{p\})$  over a point  $p \in M$  is the  $m$ -dimensional **tangent space**  $T_p M$ . The triple  $(TM, M, \pi)$  is called the **tangent bundle** of  $M$ .

We shall now equip  $TM$  with the structure of a smooth manifold. For every local chart  $x : U \rightarrow \mathbb{R}^m$  from the maximal atlas  $\hat{\mathcal{A}}$  of  $M$  we define a local chart

$$x^* : \pi^{-1}(U) \rightarrow \mathbb{R}^m \times \mathbb{R}^m$$

on the tangent bundle  $TM$  by the formula

$$x^* : \left( p, \sum_{k=1}^m v_k \left( \frac{\partial}{\partial x_k} \right)_p \right) \mapsto (x(p), (v_1, \dots, v_m)).$$

Proposition 3.19 shows that the map  $x^*$  is well-defined. The collection

$$\{(x^*)^{-1}(W) \subset TM \mid (U, x) \in \hat{\mathcal{A}} \text{ and } W \subset x(U) \times \mathbb{R}^m \text{ open}\}$$

is a basis for a topology  $\mathcal{T}_{TM}$  on  $TM$  and  $(\pi^{-1}(U), x^*)$  is a chart on the  $2m$ -dimensional **topological manifold**  $(TM, \mathcal{T}_{TM})$ .

If  $(U, x)$  and  $(V, y)$  are two charts in  $\hat{\mathcal{A}}$  such that  $p \in U \cap V$  then the transition map

$$(y^*) \circ (x^*)^{-1} : x^*(\pi^{-1}(U \cap V)) \rightarrow \mathbb{R}^m \times \mathbb{R}^m$$

is given by

$$(a, b) \mapsto (y \circ x^{-1}(a), \sum_{k=1}^m \frac{\partial y_1}{\partial x_k}(x^{-1}(a))b_k, \dots, \sum_{k=1}^m \frac{\partial y_m}{\partial x_k}(x^{-1}(a))b_k),$$

see Exercise 4.1. Since we are assuming that  $y \circ x^{-1}$  is differentiable it follows that  $(y^*) \circ (x^*)^{-1}$  is also differentiable. This means that

$$\mathcal{A}^* = \{(\pi^{-1}(U), x^*) \mid (U, x) \in \hat{\mathcal{A}}\}$$

is a  $C^r$ -atlas on  $TM$  so  $(TM, \hat{\mathcal{A}}^*)$  is a **differentiable manifold**. The surjective projection map  $\pi : TM \rightarrow M$  is clearly differentiable.

We shall now describe how the tangent bundle  $(TM, M, \pi)$  can be given the structure of an  $m$ -dimensional differentiable vector bundle. For each point  $p \in M$  the fibre  $\pi^{-1}(\{p\})$  is the tangent space  $T_p M$  and hence an  $m$ -dimensional vector space. For a local chart  $x : U \rightarrow \mathbb{R}^m$  in the maximal atlas  $\hat{\mathcal{A}}$  of  $M$  we define  $\bar{x} : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^m$  by

$$\bar{x} : \left( p, \sum_{k=1}^m v_k \left( \frac{\partial}{\partial x_k} \right)_p \right) \mapsto (p, (v_1, \dots, v_m)).$$

The restriction  $\bar{x}_p = \bar{x}|_{T_p M} : T_p M \rightarrow \{p\} \times \mathbb{R}^m$  to the tangent space  $T_p M$  is given by

$$\bar{x}_p : \sum_{k=1}^m v_k \left( \frac{\partial}{\partial x_k} \right)_p \mapsto (v_1, \dots, v_m),$$

so it is clearly a vector space isomorphism. This implies that the map

$$\bar{x} : \pi^{-1}(U) \rightarrow U \times \mathbb{R}^m$$

is a local bundle chart. If  $(U, x)$  and  $(V, y)$  are two charts in  $\hat{\mathcal{A}}$  such that  $p \in U \cap V$  then the transition map

$$(\bar{y}) \circ (\bar{x})^{-1} : (U \cap V) \times \mathbb{R}^m \rightarrow (U \cap V) \times \mathbb{R}^m$$

is given by

$$(p, b) \mapsto \left( p, \sum_{k=1}^m \frac{\partial y_1}{\partial x_k}(p) b_k, \dots, \sum_{k=1}^m \frac{\partial y_m}{\partial x_k}(p) b_k \right).$$

This shows that

$$\mathcal{B} = \{(\pi^{-1}(U), \bar{x}) \mid (U, x) \in \hat{\mathcal{A}}\}$$

is a bundle atlas turning  $(TM, M, \pi)$  into an  $m$ -dimensional topological vector bundle. It immediately follows from above that  $(TM, M, \pi)$  together with the maximal bundle atlas  $\hat{\mathcal{B}}$  defined by  $\mathcal{B}$  is a **differentiable vector bundle**.

**Definition 4.8.** Let  $M$  be a differentiable manifold, then a section  $X : M \rightarrow TM$  of the tangent bundle is called a **vector field**. The set of smooth vector fields  $X : M \rightarrow TM$  is denoted by  $C^\infty(TM)$ .

**Example 4.9.** We have seen earlier that the 3-sphere  $S^3$  in  $\mathbb{H} \cong \mathbb{C}^2$  carries a group structure  $\cdot$  given by

$$(z, w) \cdot (\alpha, \beta) = (z\alpha - w\bar{\beta}, z\beta + w\bar{\alpha}).$$

This makes  $(S^3, \cdot)$  into a Lie group with neutral element  $e = (1, 0)$ . Put  $v_1 = (i, 0)$ ,  $v_2 = (0, 1)$  and  $v_3 = (0, i)$  and for  $k = 1, 2, 3$  define the curves  $\gamma_k : \mathbb{R} \rightarrow S^3$  with

$$\gamma_k : t \mapsto \cos t \cdot (1, 0) + \sin t \cdot v_k.$$

Then  $\gamma_k(0) = e$  and  $\dot{\gamma}_k(0) = v_k$  so  $v_1, v_2, v_3$  are elements of the tangent space  $T_e S^3$ . They are linearly independent so they generate  $T_e S^3$ . The group structure on  $S^3$  can be used to extend vectors in  $T_e S^3$  to vector fields on  $S^3$  as follows. For  $p \in S^3$  let  $L_p : S^3 \rightarrow S^3$  be the left translation on  $S^3$  by  $p$  given by  $L_p : q \mapsto p \cdot q$ . Then define the vector fields  $X_1, X_2, X_3 \in C^\infty(TS^3)$  by

$$(X_k)_p = (dL_p)_e(v_k) = \frac{d}{dt}(L_p(\gamma_k(t)))|_{t=0}.$$

It is left as an exercise for the reader to show that at a point  $p = (z, w) \in S^3$  the values of  $X_k$  at  $p$  is given by

$$\begin{aligned} (X_1)_p &= (z, w) \cdot (i, 0) = (iz, -iw), \\ (X_2)_p &= (z, w) \cdot (0, 1) = (-w, z), \\ (X_3)_p &= (z, w) \cdot (0, i) = (iw, iz). \end{aligned}$$

Our next aim is to introduce the Lie bracket on the set of vector fields  $C^\infty(TM)$  on  $M$ .

**Definition 4.10.** Let  $M$  be a smooth manifold. For two vector fields  $X, Y \in C^\infty(TM)$  we define the **Lie bracket**  $[X, Y]_p : C^\infty(M) \rightarrow \mathbb{R}$  of  $X$  and  $Y$  at  $p \in M$  by

$$[X, Y]_p(f) = X_p(Y(f)) - Y_p(X(f)).$$

The next result shows that the Lie bracket  $[X, Y]_p$  actually is an element of the tangent space  $T_pM$ .

**Proposition 4.11.** *Let  $M$  be a smooth manifold,  $X, Y \in C^\infty(TM)$  be vector fields on  $M$ ,  $f, g \in C^\infty(M)$  and  $\lambda, \mu \in \mathbb{R}$ . Then*

- (i)  $[X, Y]_p(\lambda \cdot f + \mu \cdot g) = \lambda \cdot [X, Y]_p(f) + \mu \cdot [X, Y]_p(g)$ ,
- (ii)  $[X, Y]_p(f \cdot g) = [X, Y]_p(f) \cdot g(p) + f(p) \cdot [X, Y]_p(g)$ .

PROOF.

$$\begin{aligned} & [X, Y]_p(\lambda f + \mu g) \\ &= X_p(Y(\lambda f + \mu g)) - Y_p(X(\lambda f + \mu g)) \\ &= \lambda X_p(Y(f)) + \mu X_p(Y(g)) - \lambda Y_p(X(f)) - \mu Y_p(X(g)) \\ &= \lambda [X, Y]_p(f) + \mu [X, Y]_p(g). \end{aligned}$$

$$\begin{aligned} & [X, Y]_p(f \cdot g) \\ &= X_p(Y(f \cdot g)) - Y_p(X(f \cdot g)) \\ &= X_p(f \cdot Y(g) + g \cdot Y(f)) - Y_p(f \cdot X(g) + g \cdot X(f)) \\ &= X_p(f)Y_p(g) + f(p)X_p(Y(g)) + X_p(g)Y_p(f) + g(p)X_p(Y(f)) \\ &\quad - Y_p(f)X_p(g) - f(p)Y_p(X(g)) - Y_p(g)X_p(f) - g(p)Y_p(X(f)) \\ &= f(p)\{X_p(Y(g)) - Y_p(X(g))\} + g(p)\{X_p(Y(f)) - Y_p(X(f))\} \\ &= f(p)[X, Y]_p(g) + g(p)[X, Y]_p(f). \end{aligned}$$

□

Proposition 4.11 implies that if  $X, Y$  are smooth vector fields on  $M$  then the map  $[X, Y] : M \rightarrow TM$  given by  $[X, Y] : p \mapsto [X, Y]_p$  is a section of the tangent bundle. In Proposition 4.13 we shall prove that this section is smooth. For this we need the following technical lemma.

**Lemma 4.12.** *Let  $M^m$  be a smooth manifold and  $X : M \rightarrow TM$  be a section of  $TM$ . Then the following conditions are equivalent*

- (i) *the section  $X$  is smooth,*

(ii) if  $(U, x)$  is a chart on  $M$  then the functions  $a_1, \dots, a_m : U \rightarrow \mathbb{R}$  given by

$$X|_U = \sum_{k=1}^m a_k \frac{\partial}{\partial x_k},$$

are smooth,

(iii) if  $f : V \rightarrow \mathbb{R}$  defined on an open subset  $V$  of  $M$  is smooth, then the function  $X(f) : V \rightarrow \mathbb{R}$  with  $X(f)(p) = X_p(f)$  is smooth.

PROOF. (i)  $\Rightarrow$  (ii): The functions  $a_k = \pi_{m+k} \circ x^* \circ X|_U : U \rightarrow \pi^{-1}(U) \rightarrow x(U) \times \mathbb{R}^m \rightarrow \mathbb{R}$  are restrictions of compositions of smooth maps so therefore smooth.

(ii)  $\Rightarrow$  (iii): Let  $(U, x)$  be a chart on  $M$  such that  $U$  is contained in  $V$ . By assumption the map

$$X(f)|_U = \sum_{i=1}^m a_i \frac{\partial f}{\partial x_i}$$

is smooth. This is true for each such chart  $(U, x)$  so the function  $X(f)$  is smooth.

(iii)  $\Rightarrow$  (i): Note that the smoothness of the section  $X$  is equivalent to  $x^* \circ X|_U : U \rightarrow \mathbb{R}^{2m}$  being smooth for all charts  $(U, x)$  on  $M$ . On the other hand, this is equivalent to  $x_k^* = \pi_k \circ x^* \circ X|_U : U \rightarrow \mathbb{R}$  being smooth for all  $k = 1, 2, \dots, 2m$  and all charts  $(U, x)$  on  $M$ . It is trivial that the coordinates  $x_k^* = x_k$  for  $k = 1, \dots, m$  are smooth. But  $x_{m+k}^* = a_k = X(x_k)$  for  $k = 1, \dots, m$  hence also smooth by assumption.  $\square$

**Proposition 4.13.** *Let  $M$  be a manifold and  $X, Y \in C^\infty(TM)$  be vector fields on  $M$ . Then the section  $[X, Y] : M \rightarrow TM$  of the tangent bundle given by  $[X, Y] : p \mapsto [X, Y]_p$  is smooth.*

PROOF. Let  $f : M \rightarrow \mathbb{R}$  be an arbitrary smooth function on  $M$  then  $[X, Y](f) = X(Y(f)) - Y(X(f))$  is smooth so it follows from Lemma 4.12 that the section  $[X, Y]$  is smooth.  $\square$

For later use we prove the following useful result.

**Lemma 4.14.** *Let  $M$  be a smooth manifold and  $[\cdot, \cdot]$  be the Lie bracket on the tangent bundle  $TM$ . Then*

- (i)  $[X, f \cdot Y] = X(f) \cdot Y + f \cdot [X, Y]$ ,
- (ii)  $[f \cdot X, Y] = f \cdot [X, Y] - Y(f) \cdot X$

for all  $X, Y \in C^\infty(TM)$  and  $f \in C^\infty(M)$ .

PROOF. If  $g \in C^\infty(M)$ , then

$$[X, f \cdot Y](g) = X(f \cdot Y(g)) - f \cdot Y(X(g))$$

$$\begin{aligned}
&= X(f) \cdot Y(g) + f \cdot X(Y(g)) - f \cdot Y(X(g)) \\
&= (X(f) \cdot Y + f \cdot [X, Y])(g).
\end{aligned}$$

This proves the first statement and the second follows from the skew-symmetry of the Lie bracket.  $\square$

**Definition 4.15.** A real vector space  $(V, +, \cdot)$  equipped with an operation  $[\cdot, \cdot] : V \times V \rightarrow V$  is said to be a **Lie algebra** if the following relations hold

- (i)  $[\lambda X + \mu Y, Z] = \lambda[X, Z] + \mu[Y, Z]$ ,
- (ii)  $[X, Y] = -[Y, X]$ ,
- (iii)  $[X, [Y, Z]] + [Z, [X, Y]] + [Y, [Z, X]] = 0$

for all  $X, Y, Z \in V$  and  $\lambda, \mu \in \mathbb{R}$ . The equation (iii) is called the **Jacobi identity**.

**Theorem 4.16.** Let  $M$  be a smooth manifold. The vector space  $C^\infty(TM)$  of smooth vector fields on  $M$  equipped with the Lie bracket  $[\cdot, \cdot] : C^\infty(TM) \times C^\infty(TM) \rightarrow C^\infty(TM)$  is a Lie algebra.

PROOF. See exercise 4.4.  $\square$

**Definition 4.17.** If  $\phi : M \rightarrow N$  is a **surjective** map between differentiable manifolds, then two vector fields  $X \in C^\infty(TM)$  and  $\bar{X} \in C^\infty(TN)$  are said to be  **$\phi$ -related** if  $d\phi_p(X_p) = \bar{X}_{\phi(p)}$  for all  $p \in M$ . In this situation we write  $d\phi(X) = \bar{X}$ .

**Example 4.18.** Let  $S^1$  be the unit circle in the complex plane and  $\phi : S^1 \rightarrow S^1$  be the map given by  $\phi(z) = z^2$ . Note that this is surjective but not bijective. Further let  $X$  be the vector field on  $S^1$  satisfying  $X(z) = iz$ . Then

$$d\phi_z(X_z) = \frac{d}{d\theta}(\phi(ze^{i\theta}))|_{\theta=0} = \frac{d}{d\theta}((ze^{i\theta})^2)|_{\theta=0} = 2iz^2 = 2X_{\phi(z)}.$$

This shows that the vector field  $X$  is  $\phi$ -related to  $\bar{X} = 2X$ .

**Example 4.19.** Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a surjective  $C^1$ -function and  $x, y \in \mathbb{R}$  such that  $x \neq y$ ,  $f(x) = f(y)$  and  $f'(x) \neq f'(y)$ . Further let  $\gamma : \mathbb{R} \rightarrow \mathbb{R}$  be the curve with  $\gamma(t) = t$  and define the vector field  $X \in C^1(\mathbb{R})$  by  $X_t = \dot{\gamma}(t)$ . Then for each  $t \in \mathbb{R}$  we have

$$df_t(X_t) = (f \circ \gamma(t))' = f'(t).$$

If  $\bar{X} \in C^1(\mathbb{R})$  is a vector field which is  $f$ -related to  $X$  then

$$\bar{X}_{f(x)} = df_x(X_x) = f'(x) \neq f'(y) = df_y(X_y) = \bar{X}_{f(y)}.$$

This contradicts the existence of such a vector field  $\bar{X}$ .

**Proposition 4.20.** *Let  $\phi : M \rightarrow N$  be a surjective map between differentiable manifolds and  $X, Y \in C^\infty(TM)$ ,  $\bar{X}, \bar{Y} \in C^\infty(TN)$  such that  $d\phi(X) = \bar{X}$  and  $d\phi(Y) = \bar{Y}$ . Then*

$$d\phi([X, Y]) = [\bar{X}, \bar{Y}].$$

PROOF. Let  $p \in M$  and  $f : N \rightarrow \mathbb{R}$  be a smooth function, then

$$\begin{aligned} d\phi_p([X, Y]_p)(f) &= [X, Y]_p(f \circ \phi) \\ &= X_p(Y(f \circ \phi)) - Y_p(X(f \circ \phi)) \\ &= X_p(d\phi(Y)(f) \circ \phi) - Y_p(d\phi(X)(f) \circ \phi) \\ &= d\phi(X)_{\phi(p)}(d\phi(Y)(f)) - d\phi(Y)_{\phi(p)}(d\phi(X)(f)) \\ &= [\bar{X}, \bar{Y}]_{\phi(p)}(f). \end{aligned}$$

□

**Proposition 4.21.** *Let  $M$  and  $N$  be differentiable manifolds and  $\phi : M \rightarrow N$  be a diffeomorphism. If  $X, Y \in C^\infty(TM)$  are vector fields on  $M$ , then  $d\phi(X)$  is a vector field on  $N$  and the tangent map  $d\phi : C^\infty(TM) \rightarrow C^\infty(TN)$  is a Lie algebra homomorphism i.e.*

$$d\phi([X, Y]) = [d\phi(X), d\phi(Y)].$$

PROOF. The fact that  $\phi$  is bijective implies that  $d\phi(X)$  is a section of the tangent bundle. That  $d\phi(X)$  is smooth follows directly from the fact that

$$d\phi(X)(f)(\phi(p)) = X(f \circ \phi)(p).$$

The last statement is a direct consequence of Proposition 4.20. □

**Definition 4.22.** Let  $M$  be a smooth manifold. Two vector fields  $X, Y \in C^\infty(TM)$  are said to **commute** if  $[X, Y] = 0$ .

**Proposition 4.23.** *Let  $M$  be a differentiable manifold,  $(U, x)$  be local coordinates on  $M$  and*

$$\left\{ \frac{\partial}{\partial x_k} \mid k = 1, 2, \dots, m \right\}$$

*be the induced local frame for the tangent bundle  $TM$ . Then the local frame fields commute i.e.*

$$\left[ \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l} \right] = 0 \quad \text{for all } k, l = 1, \dots, m.$$

PROOF. The map  $x : U \rightarrow x(U)$  is bijective and differentiable. The vector field  $\partial/\partial x_k \in C^\infty(TU)$  is  $x$ -related to the coordinate vector field  $\partial_{e_k} \in C^\infty(Tx(U))$ . Then Proposition 4.21 implies that

$$dx\left(\left[\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_l}\right]\right) = [\partial_{e_k}, \partial_{e_l}] = 0.$$

The last equation is an immediate consequence of the following well-known fact

$$[\partial_{e_k}, \partial_{e_l}](f) = \partial_{e_k}(\partial_{e_l}(f)) - \partial_{e_l}(\partial_{e_k}(f)) = 0$$

for all  $f \in C^2(x(U))$ . □

**Definition 4.24.** A vector field  $X \in C^\infty(TG)$  is said to be **left-invariant** if for all  $p \in G$  it is  $L_p$ -related to itself i.e.

$$(dL_p)_q(X_q) = X_{pq} \quad \text{for all } p, q \in G.$$

The set of left-invariant vector fields on  $G$  is called the **Lie algebra** of  $G$  and denoted by  $\mathfrak{g}$ .

**Remark 4.25.** It should be noted that if  $e$  is the neutral element of the Lie group  $G$  and  $X \in \mathfrak{g}$  is a left-invariant vector field then

$$X_p = (dL_p)_e(X_e).$$

This shows that the value  $X_p$  of  $X$  at  $p$  is completely determined by the value  $X_e$  at  $e$ . Hence the map  $\Phi : T_eG \rightarrow \mathfrak{g}$  given by

$$\Phi : X_e \mapsto (X : p \mapsto (dL_p)_e(X_e))$$

is a vector space isomorphism. As a direct consequence we see that the Lie algebra  $\mathfrak{g}$  is a finite dimensional subspace of  $C^\infty(TG)$ .

**Proposition 4.26.** *If  $G$  is a Lie group then its Lie algebra  $\mathfrak{g}$  is a Lie subalgebra of  $C^\infty(TG)$  i.e. if  $X, Y \in \mathfrak{g}$  then  $[X, Y] \in \mathfrak{g}$ .*

**PROOF.** If  $p \in G$  then the left translation  $L_p : G \rightarrow G$  is a diffeomorphism so it follows from Proposition 4.21 that

$$dL_p([X, Y]) = [dL_p(X), dL_p(Y)] = [X, Y]$$

for all  $X, Y \in \mathfrak{g}$ . This proves that the Lie bracket  $[X, Y]$  of two left-invariant vector fields  $X, Y$  is left-invariant. □

The linear isomorphism  $\Phi : T_eG \rightarrow \mathfrak{g}$  given by

$$\Phi : X_e \mapsto (X : p \mapsto (dL_p)_e(X_e))$$

induces a natural Lie bracket  $[\cdot, \cdot] : T_eG \times T_eG \rightarrow T_eG$  on the tangent space  $T_eG$  via

$$[X_e, Y_e] = [X, Y]_e.$$

For the classical matrix Lie groups introduced in Chapter 3, we denote their Lie algebras by  $\mathfrak{gl}_m(\mathbb{R})$ ,  $\mathfrak{sl}_m(\mathbb{R})$ ,  $\mathfrak{o}(m)$ ,  $\mathfrak{so}(m)$ ,  $\mathfrak{gl}_m(\mathbb{C})$ ,  $\mathfrak{sl}_m(\mathbb{C})$ ,  $\mathfrak{u}(m)$  and  $\mathfrak{su}(m)$ , respectively.



**Proposition 4.27.** *Let  $G$  be one of the classical matrix Lie groups and  $T_e G$  be the tangent space of  $G$  at the neutral element  $e$ . Then the Lie bracket  $[\cdot, \cdot] : T_e G \times T_e G \rightarrow T_e G$  is given by*

$$[X_e, Y_e] = X_e \cdot Y_e - Y_e \cdot X_e$$

where  $\cdot$  is the usual matrix multiplication.

**PROOF.** We prove the result for the general linear group  $\mathbf{GL}_m(\mathbb{R})$ . For the other real groups the result follows from the fact that they are all subgroups of  $\mathbf{GL}_m(\mathbb{R})$ . The same proof can be used for the complex cases.

Let  $X, Y \in \mathfrak{gl}_m(\mathbb{R})$  be left-invariant vector fields,  $f : U \rightarrow \mathbb{R}$  be a function defined locally around the identity element  $e$  and  $p$  be an arbitrary point in  $U$ . Then the derivative  $X_p(f)$  is given by

$$X_p(f) = \frac{d}{dt}(f(p \cdot \text{Exp}(tX_e)))|_{t=0} = df_p(p \cdot X_e) = df_p(X_p).$$

The general linear group  $\mathbf{GL}_m(\mathbb{R})$  is an open subset of  $\mathbb{R}^{m \times m}$  so we can apply standard arguments from multi-variable calculus. The second derivative  $Y_e(X(f))$  satisfies

$$\begin{aligned} Y_e(X(f)) &= \frac{d}{dt}(X_{\text{Exp}(tY_e)}(f))|_{t=0} \\ &= \frac{d}{dt}(df_{\text{Exp}(tY_e)}(\text{Exp}(tY_e) \cdot X_e))|_{t=0} \\ &= d^2 f_e(Y_e, X_e) + df_e(Y_e \cdot X_e). \end{aligned}$$

Here  $d^2 f_e$  is the symmetric Hessian for  $f$ . As an immediate consequence obtain

$$[X, Y]_e(f) = X_e(Y(f)) - Y_e(X(f)) = df_e(X_e \cdot Y_e - Y_e \cdot X_e).$$

□

**Theorem 4.28.** *The tangent bundle  $TG$  of a Lie group  $G$  is trivial.*

**PROOF.** Let  $\{(X_1)_e, \dots, (X_m)_e\}$  be a basis for  $T_e G$  and extend each  $(X_k)_e \in T_e G$  to the left-invariant vector field  $X_k \in \mathfrak{g}$  with

$$(X_k)_p = (dL_p)_e((X_k)_e).$$

For each  $p \in G$  the left translation  $L_p : G \rightarrow G$  is a diffeomorphism so the set  $\{(X_1)_p, \dots, (X_m)_p\}$  is a basis for the tangent space  $T_p G$ . This means that the map  $\psi : TG \rightarrow G \times \mathbb{R}^m$  given by

$$\psi : (p, \sum_{k=1}^m v_k \cdot (X_k)_p) \mapsto (p, (v_1, \dots, v_m))$$

is well-defined. It is a global bundle chart so the tangent bundle  $TG$  is trivial.  $\square$

## Exercises

**Exercise 4.1.** Let  $(M, \hat{\mathcal{A}})$  be a smooth manifold,  $(U, x)$ ,  $(V, y)$  be local charts such that  $U \cap V$  is non-empty and

$$f = y \circ x^{-1} : x(U \cap V) \rightarrow \mathbb{R}^m$$

be the corresponding transition map. Show that the local frames

$$\left\{ \frac{\partial}{\partial x_i} \mid i = 1, \dots, m \right\} \quad \text{and} \quad \left\{ \frac{\partial}{\partial y_j} \mid j = 1, \dots, m \right\}$$

for  $TM$  on  $U \cap V$  satisfy

$$\frac{\partial}{\partial x_i} = \sum_{j=1}^m \frac{\partial(f_j \circ x)}{\partial x_i} \cdot \frac{\partial}{\partial y_j}.$$

**Exercise 4.2.** Let  $\mathbf{SO}(m)$  be the special orthogonal group.

- (i) Find a basis for the tangent space  $T_e \mathbf{SO}(m)$ ,
- (ii) construct a non-vanishing vector field  $Z \in C^\infty(T\mathbf{SO}(m))$ ,
- (iii) determine all smooth vector fields on  $\mathbf{SO}(2)$ .

**The Hairy Ball Theorem.** There does not exist a continuous non-vanishing vector field  $X \in C^0(TS^{2m})$  on the even dimensional sphere  $S^{2m}$ .

**Exercise 4.3.** Employ the Hairy Ball Theorem to show that the tangent bundle  $TS^{2m}$  is not trivial. Then construct a non-vanishing vector field  $X \in C^\infty(TS^{2m+1})$  on the odd-dimensional sphere  $S^{2m+1}$ .

**Exercise 4.4.** Find a proof of Theorem 4.16.

**Exercise 4.5.** The Lie algebra  $\mathfrak{sl}_2(\mathbb{R})$  of the special linear group  $\mathbf{SL}_2(\mathbb{R})$  is generated by

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Show that the Lie brackets of  $\mathfrak{sl}_2(\mathbb{R})$  satisfy

$$[X, Y] = 2Z, \quad [Z, X] = 2Y, \quad [Y, Z] = -2X.$$

**Exercise 4.6.** The Lie algebra  $\mathfrak{su}(2)$  of the special unitary group  $\mathbf{SU}(2)$  is generated by

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad Z = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Show that the corresponding Lie bracket relations are given by

$$[X, Y] = 2Z, \quad [Z, X] = 2Y, \quad [Y, Z] = 2X.$$



## Riemannian Manifolds

In this chapter we introduce the notion of a Riemannian manifold  $(M, g)$ . The metric  $g$  provides us with an inner product on each tangent space and can be used to measure angles and the lengths of curves in the manifold. This defines a distance function and turns the manifold into a metric space in a natural way. The Riemannian metric on a differentiable manifold is an important example of what is called a tensor field.

Let  $M$  be a smooth manifold,  $C^\infty(M)$  denote the commutative ring of smooth functions on  $M$  and  $C^\infty(TM)$  be the set of smooth vector fields on  $M$  forming a module over  $C^\infty(M)$ . Put

$$C_0^\infty(TM) = C^\infty(M)$$

and for each positive integer  $r$  let

$$C_r^\infty(TM) = C^\infty(TM) \otimes \cdots \otimes C^\infty(TM)$$

be the  $r$ -fold tensor product of  $C^\infty(TM)$  over  $C^\infty(M)$ .

**Definition 5.1.** Let  $M$  be a differentiable manifold. A smooth **tensor field**  $A$  on  $M$  of type  $(r, s)$  is a map  $A : C_r^\infty(TM) \rightarrow C_s^\infty(TM)$  which is multi-linear over  $C^\infty(M)$  i.e. satisfying

$$\begin{aligned} & A(X_1 \otimes \cdots \otimes X_{k-1} \otimes (f \cdot Y + g \cdot Z) \otimes X_{k+1} \otimes \cdots \otimes X_r) \\ &= f \cdot A(X_1 \otimes \cdots \otimes X_{k-1} \otimes Y \otimes X_{k+1} \otimes \cdots \otimes X_r) \\ & \quad + g \cdot A(X_1 \otimes \cdots \otimes X_{k-1} \otimes Z \otimes X_{k+1} \otimes \cdots \otimes X_r) \end{aligned}$$

for all  $X_1, \dots, X_r, Y, Z \in C^\infty(TM)$ ,  $f, g \in C^\infty(M)$  and  $k = 1, \dots, r$ . For the rest of this work we shall for  $A(X_1 \otimes \cdots \otimes X_r)$  use the notation

$$A(X_1, \dots, X_r).$$

The next general result provides us with the most important property of a tensor field. It shows that the value of  $A(X_1, \dots, X_r)$  at a point  $p \in M$  only depends on the values of the vector fields  $X_1, \dots, X_r$  at  $p$  and is independent of their values away from  $p$ .

**Proposition 5.2.** *Let  $A : C_r^\infty(TM) \rightarrow C_s^\infty(TM)$  be a tensor field of type  $(r, s)$  and  $p \in M$ . Let  $X_1, \dots, X_r$  and  $Y_1, \dots, Y_r$  be smooth vector fields on  $M$  such that  $(X_k)_p = (Y_k)_p$  for each  $k = 1, \dots, r$ . Then*

$$A(X_1, \dots, X_r)(p) = A(Y_1, \dots, Y_r)(p).$$

PROOF. We shall prove the statement for  $r = 1$ , the rest follows by induction. Put  $X = X_1$  and  $Y = Y_1$  and let  $(U, x)$  be local coordinates on  $M$ . Choose a function  $f \in C^\infty(M)$  such that  $f(p) = 1$ ,

$$\text{support}(f) = \overline{\{p \in M \mid f(p) \neq 0\}}$$

is contained in  $U$  and define the vector fields  $v_1, \dots, v_m \in C^\infty(TM)$  on  $M$  by

$$(v_k)_q = \begin{cases} f(q) \cdot \left(\frac{\partial}{\partial x_k}\right)_q & \text{if } q \in U \\ 0 & \text{if } q \notin U \end{cases}$$

Then there exist functions  $\rho_k, \sigma_k \in C^\infty(M)$  such that

$$f \cdot X = \sum_{k=1}^m \rho_k \cdot v_k \quad \text{and} \quad f \cdot Y = \sum_{k=1}^m \sigma_k \cdot v_k.$$

This implies that

$$\begin{aligned} A(X)(p) &= f(p)A(X)(p) \\ &= (f \cdot A(X))(p) \\ &= A(f \cdot X)(p) \\ &= A\left(\sum_{k=1}^m \rho_k \cdot v_k\right)(p) \\ &= \sum_{k=1}^m (\rho_k \cdot A(v_k))(p) \\ &= \sum_{k=1}^m \rho_k(p)A(v_k)(p) \end{aligned}$$

and similarly

$$A(Y)(p) = \sum_{k=1}^m \sigma_k(p)A(v_k)(p).$$

The fact that  $X_p = Y_p$  shows that  $\rho_k(p) = \sigma_k(p)$  for all  $k$ . As a direct consequence we see that

$$A(X)(p) = A(Y)(p).$$

□

For a tensor  $A$  we shall by  $A_p$  denote the multi-linear restriction of  $A$  to the  $r$ -fold tensor product  $T_p M \otimes \cdots \otimes T_p M$  of the vector space  $T_p M$  over  $\mathbb{R}$  given by

$$A_p : ((X_1)_p, \dots, (X_r)_p) \mapsto A(X_1, \dots, X_r)(p).$$

**Definition 5.3.** Let  $M$  be a smooth manifold. A **Riemannian metric**  $g$  on  $M$  is a tensor field  $g : C_2^\infty(TM) \rightarrow C_0^\infty(TM)$  such that for each  $p \in M$  the restriction  $g_p$  of  $g$  to  $T_p M \otimes T_p M$  with

$$g_p : (X_p, Y_p) \mapsto g(X, Y)(p)$$

is an inner product on the tangent space  $T_p M$ . The pair  $(M, g)$  is called a **Riemannian manifold**. The study of Riemannian manifolds is called Riemannian Geometry. Geometric properties of  $(M, g)$  which only depend on the metric  $g$  are said to be **intrinsic** or metric properties.

**Definition 5.4.** Let  $\gamma : I \rightarrow M$  be a  $C^1$ -curve in  $M$ . Then the **length**  $L(\gamma)$  of  $\gamma$  is defined by

$$L(\gamma) = \int_I \sqrt{g(\dot{\gamma}(t), \dot{\gamma}(t))} dt.$$

**Example 5.5.** The standard inner product on the vector space  $\mathbb{R}^m$  given by

$$\langle X, Y \rangle_{\mathbb{R}^m} = X^t \cdot Y = \sum_{k=1}^m X_k Y_k$$

defines a Riemannian metric on  $\mathbb{R}^m$ . The Riemannian manifold

$$E^m = (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$$

is called the  $m$ -dimensional **Euclidean space**.

**Example 5.6.** Equip the vector space  $\mathbb{R}^m$  with the Riemannian metric  $g$  given by

$$g_p(X, Y) = \frac{4}{(1 + |p|_{\mathbb{R}^m}^2)^2} \langle X, Y \rangle_{\mathbb{R}^m}.$$

The Riemannian manifold  $\Sigma^m = (\mathbb{R}^m, g)$  is called the  $m$ -dimensional **punctured round sphere**. Let  $\gamma : \mathbb{R}^+ \rightarrow \Sigma^m$  be the curve with

$$\gamma : t \mapsto (t, 0, \dots, 0).$$

Then the length  $L(\gamma)$  of  $\gamma$  can be determined as follows

$$L(\gamma) = 2 \int_0^\infty \frac{\sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle}}{1 + |\gamma|^2} dt = 2 \int_0^\infty \frac{dt}{1 + t^2} = 2[\arctan(t)]_0^\infty = \pi.$$

**Example 5.7.** Let  $B_1^m(0)$  be the open unit ball in  $\mathbb{R}^m$  given by

$$B_1^m(0) = \{p \in \mathbb{R}^m \mid |p|_{\mathbb{R}^m} < 1\}.$$

By the  $m$ -dimensional **hyperbolic ball** we mean  $B_1^m(0)$  equipped with the Riemannian metric

$$g_p(X, Y) = \frac{4}{(1 - |p|_{\mathbb{R}^m}^2)^2} \langle X, Y \rangle_{\mathbb{R}^m}.$$

Let  $\gamma : (0, 1) \rightarrow B_1^m(0)$  be a curve given by  $\gamma : t \mapsto (t, 0, \dots, 0)$ . Then

$$L(\gamma) = 2 \int_0^1 \frac{\sqrt{\langle \dot{\gamma}, \dot{\gamma} \rangle}}{1 - |\gamma|^2} dt = 2 \int_0^1 \frac{dt}{1 - t^2} = [\log(\frac{1+t}{1-t})]_0^1 = \infty$$

As we shall now see a Riemannian manifold  $(M, g)$  has the structure of a metric space  $(M, d)$  in a natural way.

**Proposition 5.8.** *Let  $(M, g)$  be a Riemannian manifold which is path-connected. For two points  $p, q \in M$  let  $C_{pq}$  denote the set of  $C^1$ -curves  $\gamma : [0, 1] \rightarrow M$  such that  $\gamma(0) = p$  and  $\gamma(1) = q$  and define the function  $d : M \times M \rightarrow \mathbb{R}_0^+$  by*

$$d(p, q) = \inf\{L(\gamma) \mid \gamma \in C_{pq}\}.$$

Then  $(M, d)$  is a **metric space** i.e. for all  $p, q, r \in M$  we have

- (i)  $d(p, q) \geq 0$ ,
- (ii)  $d(p, q) = 0$  if and only if  $p = q$ ,
- (iii)  $d(p, q) = d(q, p)$ ,
- (iv)  $d(p, q) \leq d(p, r) + d(r, q)$ .

The topology on  $M$  induced by the metric  $d$  is identical to the one  $M$  carries as a topological manifold  $(M, \mathcal{T})$ , see Definition 2.1.

PROOF. See for example: Peter Petersen, *Riemannian Geometry*, Graduate Texts in Mathematics **171**, Springer (1998).  $\square$

A Riemannian metric on a differentiable manifold induces a Riemannian metric on any of its submanifolds as follows.

**Definition 5.9.** Let  $M$  be a submanifold of Riemannian manifold  $(N, h)$ . Then the smooth tensor field  $g : C_2^\infty(TM) \rightarrow C_0^\infty(M)$  given by

$$g(X, Y) : p \mapsto h_p(X_p, Y_p)$$

is a Riemannian metric on  $M$  called the **induced metric** on  $M$  in  $(N, h)$ .

**Example 5.10.** The Euclidean metric  $\langle \cdot, \cdot \rangle_{\mathbb{R}^n}$  on  $\mathbb{R}^n$  induces Riemannian metrics on the following submanifolds.

- (i) the  $m$ -dimensional sphere  $S^m \subset \mathbb{R}^n$ , with  $n = m + 1$ ,



- (ii) the tangent bundle  $TS^m \subset \mathbb{R}^n$ , where  $n = 2(m + 1)$ ,
- (iii) the  $m$ -dimensional torus  $T^m \subset \mathbb{R}^n$ , with  $n = 2m$ ,
- (iv) the  $m$ -dimensional real projective space  $\mathbb{R}P^m$  as a subset of  $\text{Sym}(\mathbb{R}^{m+1}) \cong \mathbb{R}^n$ , where  $n = (m + 2)(m + 1)/2$ .

**Example 5.11.** The vector space  $\mathbb{C}^{m \times m}$  of complex  $m \times m$  matrices carries a natural Riemannian metric  $g$  given by

$$g(Z, W) = \text{Re}(\text{trace}(\bar{Z}^t W)) \quad \text{for all } Z, W \in \mathbb{C}^{m \times m}.$$

This induces metrics on the submanifolds of  $\mathbb{C}^{m \times m}$  such as  $\mathbb{R}^{m \times m}$  and the classical Lie groups  $\mathbf{GL}_m(\mathbb{R})$ ,  $\mathbf{SL}_m(\mathbb{R})$ ,  $\mathbf{O}(m)$ ,  $\mathbf{SO}(m)$ ,  $\mathbf{GL}_m(\mathbb{C})$ ,  $\mathbf{SL}_m(\mathbb{C})$ ,  $\mathbf{U}(m)$ ,  $\mathbf{SU}(m)$ .

Our next important step is to prove that every differentiable manifold  $M$  can be equipped with a Riemannian metric  $g$ . For this we need the following fact from topology.

**Fact 5.12.** *Every locally compact Hausdorff space with countable basis is paracompact i.e. every open cover has an open refinement that is locally finite.*

**Corollary 5.13.** *Let  $(M, \mathcal{T})$  be a topological manifold. Let the collection  $(U_\alpha)_{\alpha \in I}$  be an open covering of  $M$  such that for each  $\alpha \in I$  the pair  $(U_\alpha, \phi_\alpha)$  is a chart on  $M$ . Then there exists*

- (i) *a locally finite open refinement  $(W_\beta)_{\beta \in J}$  such that for all  $\beta \in J$ ,  $W_\beta$  is an open neighbourhood for a chart  $(W_\beta, \phi_\beta)$ , and*
- (ii) *a partition of unity  $(f_\beta)_{\beta \in J}$  such that  $\text{support}(f_\beta) \subset W_\beta$ .*

**Theorem 5.14.** *Let  $(M^m, \hat{\mathcal{A}})$  be a differentiable manifold. Then there exists a Riemannian metric  $g$  on  $M$ .*

PROOF. For each point  $p \in M$  let  $(U_p, \phi_p) \in \hat{\mathcal{A}}$  be a chart such that  $p \in U_p$ . Then  $(U_p)_{p \in M}$  is an open covering as in Corollary 5.13. Let  $(W_\beta)_{\beta \in J}$  be a locally finite open refinement,  $(W_\beta, x^\beta)$  be charts on  $M$  and  $(f_\beta)_{\beta \in J}$  be a partition of unity such that  $\text{support}(f_\beta)$  is contained in  $W_\beta$ . Let  $\langle \cdot, \cdot \rangle_{\mathbb{R}^m}$  be the Euclidean metric on  $\mathbb{R}^m$ . Then for  $\beta \in J$  define  $g_\beta : C_2^\infty(TM) \rightarrow C_0^\infty(TM)$  by

$$g_\beta\left(\frac{\partial}{\partial x_k^\beta}, \frac{\partial}{\partial x_l^\beta}\right)(p) = \begin{cases} f_\beta(p) \cdot \langle e_k, e_l \rangle_{\mathbb{R}^m} & \text{if } p \in W_\beta \\ 0 & \text{if } p \notin W_\beta \end{cases}$$

Note that at each point only finitely many of  $g_\beta$  are non-zero. This means that the well-defined tensor  $g : C_2^\infty(TM) \rightarrow C_0^\infty(TM)$  given by

$$g = \sum_{\beta \in J} g_\beta$$

is a Riemannian metric on  $M$ . □

**Definition 5.15.** Let  $(M, g)$  and  $(N, h)$  be Riemannian manifolds. A map  $\phi : (M, g) \rightarrow (N, h)$  is said to be **conformal** if there exists a function  $\lambda : M \rightarrow \mathbb{R}$  such that

$$e^{\lambda(p)} g_p(X_p, Y_p) = h_{\phi(p)}(d\phi_p(X_p), d\phi_p(Y_p)),$$

for all  $X, Y \in C^\infty(TM)$  and  $p \in M$ . The function  $e^\lambda$  is called the **conformal factor** of  $\phi$ . A conformal map with  $\lambda \equiv 0$  is said to be **isometric**. An isometric diffeomorphism is called an **isometry**.

**Definition 5.16.** For a Riemannian manifold  $(M, g)$  we denote by  $\mathcal{I}(M)$  the set of all its isometries. If  $\phi, \psi \in \mathcal{I}(M)$  then it is clear that the composition  $\psi \circ \phi$  and the inverse  $\phi^{-1}$  are also isometries. The pair  $(\mathcal{I}(M), \circ)$  is called the **isometry group** of  $(M, g)$ . The operation is clearly associative and the identity map is its neutral element.

**Definition 5.17.** The isometry group  $\mathcal{I}(M)$  of a Riemannian manifold  $(M, g)$  is said to be **transitive** if for all  $p, q \in M$  there exists an isometry  $\phi_{pq} : M \rightarrow M$  such that  $\phi_{pq}(p) = q$ . In that case  $(M, g)$  is called a **Riemannian homogeneous space**.

**Example 5.18.** On the standard unit sphere  $S^m$  we have an action  $\mathbf{O}(m+1) \times S^m \rightarrow S^m$  of the orthogonal group  $\mathbf{O}(m+1)$  given by

$$(p, x) \mapsto p \cdot x$$

where  $\cdot$  is the standard matrix multiplication. The following shows that the  $\mathbf{O}(m+1)$ -action on  $S^m$  is isometric

$$\langle pX, pY \rangle = X^t p^t p Y = X^t Y = \langle X, Y \rangle.$$

This means that the orthogonal group  $\mathbf{O}(m+1)$  is a subgroup of the isometry group  $\mathcal{I}(S^m)$ . It is easily seen that  $\mathbf{O}(m+1)$  acts transitively on the sphere  $S^m$  so it is a homogeneous space.

**Example 5.19.** The standard inner product on  $\mathbb{R}^{m \times m}$  induces a Riemannian metric on the orthogonal group  $\mathbf{O}(m)$ , given by

$$\langle X, Y \rangle = \text{trace}(X^t Y).$$

Applying the left translation  $L_p : \mathbf{O}(m) \rightarrow \mathbf{O}(m)$  with  $L_p : q \mapsto pq$  we see that the tangent space  $T_p \mathbf{O}(m)$  of  $\mathbf{O}(m)$  at  $p$  is

$$T_p \mathbf{O}(m) = \{pX \mid X^t + X = 0\}.$$

The differential  $(dL_p)_q : T_q \mathbf{O}(m) \rightarrow T_{pq} \mathbf{O}(m)$  of  $L_p$  at  $q \in \mathbf{O}(m)$  satisfies

$$(dL_p)_q : qX \mapsto pqX.$$

We then have

$$g_{pq}((dL_p)_q(qX), (dL_p)_q(qY)) = \text{trace}((pqX)^t pqY)$$

$$\begin{aligned}
&= \text{trace}(X^t q^t p^t p q Y) \\
&= \text{trace}(q X)^t (q Y). \\
&= g_q(q X, q Y).
\end{aligned}$$

This shows that the left translation  $L_p : \mathbf{O}(m) \rightarrow \mathbf{O}(m)$  is an isometry for each  $p \in \mathbf{O}(m)$ .

**Definition 5.20.** A Riemannian metric  $g$  on a Lie group  $G$  is said to be **left-invariant** if for each  $p \in G$  the left translation  $L_p : G \rightarrow G$  is an isometry. A Lie group  $(G, g)$  with a left-invariant metric is called a **Riemannian Lie group**.

**Remark 5.21.** It should be noted that if  $(G, g)$  is a Riemannian Lie group and  $X, Y \in \mathfrak{g}$  are left-invariant vector fields then

$$g_p(X_p, Y_p) = g_p((dL_p)_e(X_e), (dL_p)_e(Y_e)) = g_e(X_e, Y_e).$$

This tells us that a left-invariant metric  $g$  on  $G$  is completely determined by the inner product  $g_e : T_e G \times T_e G \rightarrow \mathbb{R}$  on the tangent space at the neutral element  $e \in G$ .

**Theorem 5.22.** *A Riemannian Lie group  $(G, g)$  is a Riemannian homogeneous space.*

**PROOF.** For arbitrary elements  $p, q \in G$  the left-translation  $\phi_{pq} = L_{qp^{-1}}$  by  $pq^{-1} \in G$  is an isometry satisfying  $\phi_{pq}(p) = q$ . This shows that the isometry group  $\mathcal{I}(G)$  is transitive.  $\square$

We shall now equip the real projective space  $\mathbb{R}P^m$  with a Riemannian metric.

**Example 5.23.** Let  $S^m$  be the unit sphere in  $\mathbb{R}^{m+1}$  and  $\text{Sym}(\mathbb{R}^{m+1})$  be the linear space of symmetric real  $(m+1) \times (m+1)$  matrices equipped with the metric  $g$  given by

$$g(X, Y) = \frac{1}{8} \text{trace}(X^t \cdot Y).$$

As in Example 3.24 we define a map  $\phi : S^m \rightarrow \text{Sym}(\mathbb{R}^{m+1})$  by

$$\phi : p \mapsto (\rho_p : q \mapsto 2\langle q, p \rangle p - q).$$

Let  $\alpha, \beta : \mathbb{R} \rightarrow S^m$  be two curves such that  $\alpha(0) = p = \beta(0)$  and put  $X = \dot{\alpha}(0)$ ,  $Y = \dot{\beta}(0)$ . Then for  $\gamma \in \{\alpha, \beta\}$  we have

$$d\phi_p(\dot{\gamma}(0)) = (q \mapsto 2\langle q, \dot{\gamma}(0) \rangle p + 2\langle q, p \rangle \dot{\gamma}(0)).$$

If  $\mathcal{B}$  is an orthonormal basis for  $\mathbb{R}^{m+1}$ , then

$$g(d\phi_p(X), d\phi_p(Y))$$

$$\begin{aligned}
&= \frac{1}{8} \text{trace}(d\phi_p(X)^t \cdot d\phi_p(Y)) \\
&= \frac{1}{8} \sum_{q \in \mathcal{B}} \langle q, d\phi_p(X)^t \cdot d\phi_p(Y)q \rangle \\
&= \frac{1}{8} \sum_{q \in \mathcal{B}} \langle d\phi_p(X)q, d\phi_p(Y)q \rangle \\
&= \frac{1}{2} \sum_{q \in \mathcal{B}} \langle \langle q, X \rangle p + \langle q, p \rangle X, \langle q, Y \rangle p + \langle q, p \rangle Y \rangle \\
&= \frac{1}{2} \sum_{q \in \mathcal{B}} \{ \langle p, p \rangle \langle X, q \rangle \langle q, Y \rangle + \langle X, Y \rangle \langle p, q \rangle \langle p, q \rangle \} \\
&= \frac{1}{2} \{ \langle X, Y \rangle + \langle X, Y \rangle \} \\
&= \langle X, Y \rangle.
\end{aligned}$$

This proves that the immersion  $\phi$  is isometric. In Example 3.24 we have seen that the image  $\phi(S^m)$  can be identified with the real projective space  $\mathbb{R}P^m$ . This inherits the induced metric from  $\mathbb{R}^{(m+1) \times (m+1)}$  and the map  $\phi : S^m \rightarrow \mathbb{R}P^m$  is what is called an isometric double cover of  $\mathbb{R}P^m$ .

Long before John Nash became famous in Hollywood he proved the next remarkable result in his paper *The embedding problem for Riemannian manifolds*, Ann. of Math. **63** (1956), 20-63. It implies that every Riemannian manifold can be realized as a submanifold of a Euclidean space. The original proof of Nash was later simplified, see for example Matthias Gunther, *On the perturbation problem associated to isometric embeddings of Riemannian manifolds*, Annals of Global Analysis and Geometry **7** (1989), 69-77.

**Deep Result 5.24.** *For  $3 \leq r \leq \infty$  let  $(M, g)$  be a Riemannian  $C^r$ -manifold. Then there exists an isometric  $C^r$ -embedding of  $(M, g)$  into a Euclidean space  $\mathbb{R}^n$ .*

We shall now see that parametrizations can be very useful tools for studying the intrinsic geometry of a Riemannian manifold  $(M, g)$ . Let  $p$  be a point of  $M$  and  $\hat{\psi} : U \rightarrow M$  be a local parametrization of  $M$  with  $q \in U$  and  $\hat{\psi}(q) = p$ . The differential  $d\hat{\psi}_q : T_q\mathbb{R}^m \rightarrow T_pM$  is bijective so there exist neighbourhoods  $U_q$  of  $q$  and  $U_p$  of  $p$  such that the restriction  $\psi = \hat{\psi}|_{U_q} : U_q \rightarrow U_p$  is a diffeomorphism. On  $U_q$  we have the canonical frame  $\{e_1, \dots, e_m\}$  for  $TU_q$  so  $\{d\psi(e_1), \dots, d\psi(e_m)\}$  is a local frame for  $TM$  over  $U_p$ . We then define the pull-back metric

$\tilde{g} = \psi^*g$  on  $U_q$  by

$$\tilde{g}(e_k, e_l) = g(d\psi(e_k), d\psi(e_l)).$$

Then  $\psi : (U_q, \tilde{g}) \rightarrow (U_p, g)$  is an isometry so the intrinsic geometry of  $(U_q, \tilde{g})$  and that of  $(U_p, g)$  are exactly the same.

**Example 5.25.** Let  $G$  be one of the classical Lie groups and  $e$  be the neutral element of  $G$ . Let  $\{X_1, \dots, X_m\}$  be a basis for the Lie algebra  $\mathfrak{g}$  of  $G$ . For  $p \in G$  define  $\psi_p : \mathbb{R}^m \rightarrow G$  by

$$\psi_p : (t_1, \dots, t_m) \mapsto L_p\left(\prod_{k=1}^m \text{Exp}(t_k X_k(e))\right)$$

where  $L_p : G \rightarrow G$  is the left translation given by  $L_p(q) = pq$ . Then

$$(d\psi_p)_0(e_k) = X_k(p)$$

for all  $k$ . This means that the differential  $(d\psi_p)_0 : T_0\mathbb{R}^m \rightarrow T_pG$  is an isomorphism so there exist open neighbourhoods  $U_0$  of 0 and  $U_p$  of  $p$  such that the restriction of  $\psi$  to  $U_0$  is bijective onto its image  $U_p$  and hence a local parametrization of  $G$  around  $p$ .

We shall now study the normal bundle of a submanifold of a given Riemannian manifold. This is an important example of the notion of a vector bundle over a manifold.

**Definition 5.26.** Let  $M$  be a submanifold of the Riemannian manifold  $(N, h)$ . For a point  $p \in M$  we define the **normal space**  $N_pM$  of  $M$  at  $p$  by

$$N_pM = \{X \in T_pN \mid h_p(X, Y) = 0 \text{ for all } Y \in T_pM\}.$$

For all  $p \in M$  we have the orthogonal decomposition

$$T_pN = T_pM \oplus N_pM.$$

The **normal bundle** of  $M$  in  $N$  is defined by

$$NM = \{(p, X) \mid p \in M, X \in N_pM\}.$$

**Example 5.27.** Let  $S^m$  be the unit sphere in  $\mathbb{R}^{m+1}$  equipped with its standard Euclidean metric  $\langle \cdot, \cdot \rangle$ . If  $p \in S^m$  then the tangent space  $T_pS^m$  of  $S^m$  at  $p$  is

$$T_pS^m = \{X \in \mathbb{R}^{m+1} \mid \langle p, X \rangle = 0\}$$

so the normal space  $N_pS^m$  of  $S^m$  at  $p$  satisfies

$$N_pS^m = \{\lambda p \in \mathbb{R}^{m+1} \mid \lambda \in \mathbb{R}\}.$$

This shows that the normal bundle  $NS^m$  of  $S^m$  in  $\mathbb{R}^{m+1}$  is given by

$$NS^m = \{(p, \lambda p) \in \mathbb{R}^{2m+2} \mid p \in S^m, \lambda \in \mathbb{R}\}.$$

**Theorem 5.28.** *Let  $M^m$  be a smooth submanifold of the Riemannian manifold  $(N^n, h)$ . Then the normal bundle  $(NM, M, \pi)$  is a smooth  $(n - m)$ -dimensional vector bundle over  $M$ .*

PROOF. See Exercise 5.6. □

We shall now determine the normal bundle  $N\mathbf{O}(m)$  of the orthogonal group  $\mathbf{O}(m)$  as a submanifold of  $\mathbb{R}^{m \times m}$ .

**Example 5.29.** The standard Euclidean scalar product on the space  $\mathbb{R}^{m \times m}$  induces a left-invariant Riemannian metric on  $\mathbf{O}(m)$  as a subset. This satisfies

$$\langle X, Y \rangle = \text{trace}(X^t Y).$$

As we have already seen, the tangent space  $T_e \mathbf{O}(m)$  of  $\mathbf{O}(m)$  at the neutral element  $e$  is

$$T_e \mathbf{O}(m) = \{X \in \mathbb{R}^{m \times m} \mid X^t + X = 0\}$$

and the tangent bundle  $T\mathbf{O}(m)$  of  $\mathbf{O}(m)$  is given by

$$T\mathbf{O}(m) = \{(p, pX) \mid p \in \mathbf{O}(m), X \in T_e \mathbf{O}(m)\}.$$

The space  $\mathbb{R}^{m \times m}$  of real  $m \times m$  matrices has a linear decomposition

$$\mathbb{R}^{m \times m} = \text{Sym}(\mathbb{R}^m) \oplus T_e \mathbf{O}(m)$$

and every element  $X \in \mathbb{R}^{m \times m}$  can be decomposed  $X = X^\top + X^\perp$  in its symmetric and skew-symmetric parts given by

$$X^\top = \frac{1}{2}(X + X^t) \quad \text{and} \quad X^\perp = \frac{1}{2}(X - X^t).$$

If  $X \in T_e \mathbf{O}(m)$  and  $Y \in \text{Sym}(\mathbb{R}^m)$  then

$$\begin{aligned} \langle X, Y \rangle &= \text{trace}(X^t Y) \\ &= \text{trace}(Y^t X) \\ &= \text{trace}(XY^t) \\ &= \text{trace}(-X^t Y) \\ &= -\langle X, Y \rangle. \end{aligned}$$

This means that the normal bundle  $N\mathbf{O}(m)$  of  $\mathbf{O}(m)$  in  $\mathbb{R}^{m \times m}$  is given by

$$N\mathbf{O}(m) = \{(p, pY) \mid p \in \mathbf{O}(m), Y \in \text{Sym}(\mathbb{R}^m)\}.$$

A given Riemannian metric  $g$  on a manifold  $M$  can be used to construct a family of natural metrics on the tangent bundle  $TM$  of  $M$ . The best known such examples are the Sasaki and Cheeger-Gromoll metrics. For a detailed survey on the geometry of tangent bundles

equipped with these metrics we recommend the paper: S. Gudmundson, E. Kappos, *On the geometry of tangent bundles*, Expo. Math. **20** (2002), 1-41.

## Exercises

**Exercise 5.1.** Equip  $\mathbb{R}^m$  and  $\mathbb{C}^m$  with their standard Euclidean scalar products and let  $\phi : \mathbb{R}^m \rightarrow T^m$  be the standard parametrization of the  $m$ -dimensional torus in  $\mathbb{C}^m$  satisfying  $\phi : (x_1, \dots, x_m) \mapsto (e^{ix_1}, \dots, e^{ix_m})$ . Show that  $\phi$  is isometric.

**Exercise 5.2.** The **stereographic projection** from the north pole of the  $m$ -dimensional sphere

$$\phi : (S^m - \{(1, 0, \dots, 0)\}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{m+1}}) \rightarrow (\mathbb{R}^m, \frac{4}{(1 + |x|^2)^2} \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$$

is given by

$$\phi : (x_0, \dots, x_m) \mapsto \frac{1}{1 - x_0} (x_1, \dots, x_m).$$

Show that  $\phi$  is an isometry.

**Exercise 5.3.** Let  $B_1^2(0)$  be the open unit disk in the complex plane equipped with the hyperbolic metric

$$g(X, Y) = \frac{4}{(1 - |z|^2)^2} \langle X, Y \rangle_{\mathbb{R}^2}.$$

Equip the upper half plane  $\{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$  with the Riemannian metric

$$g(X, Y) = \frac{1}{\text{Im}(z)^2} \langle X, Y \rangle_{\mathbb{R}^2}.$$

Prove that the holomorphic function  $f : B_1^2(0) \rightarrow \{z \in \mathbb{C} \mid \text{Im}(z) > 0\}$  given by

$$f : z \mapsto \frac{i + z}{1 + iz}$$

is an isometry.

**Exercise 5.4.** Equip the unitary group  $\mathbf{U}(m)$  with the Riemannian metric  $g$  given by

$$g(Z, W) = \text{Re}(\text{trace}(\bar{Z}^t W)).$$

Show that for each  $p \in \mathbf{U}(m)$  the left translation  $L_p : \mathbf{U}(m) \rightarrow \mathbf{U}(m)$  is an isometry.

**Exercise 5.5.** For the general linear group  $\mathbf{GL}_m(\mathbb{R})$  we have two Riemannian metrics  $g$  and  $h$  satisfying

$$g_p(pZ, pW) = \text{trace}((pZ)^t pW), \quad h_p(pZ, pW) = \text{trace}(Z^t W).$$

Further let  $\hat{g}, \hat{h}$  be their induced metrics on the special linear group  $\mathbf{SL}_m(\mathbb{R})$  as a subset of  $\mathbf{GL}_m(\mathbb{R})$ .



- (i) Which of the metrics  $g, h, \hat{g}, \hat{h}$  are left-invariant?
- (ii) Determine the normal space  $N_e \mathbf{SL}_m(\mathbb{R})$  of  $\mathbf{SL}_m(\mathbb{R})$  in  $\mathbf{GL}_m(\mathbb{R})$  with respect to  $g$
- (iii) Determine the normal bundle  $N\mathbf{SL}_m(\mathbb{R})$  of  $\mathbf{SL}_m(\mathbb{R})$  in  $\mathbf{GL}_m(\mathbb{R})$  with respect to  $h$ .

**Exercise 5.6.** Find a proof of Theorem 5.28.



## The Levi-Civita Connection

In this chapter we introduce the Levi-Civita connection  $\nabla$  of a Riemannian manifold  $(M, g)$ . This is the most important example of the general notion of a connection in a smooth vector bundle. We deduce an explicit formula for the Levi-Civita connection for Lie groups equipped with left-invariant metrics. We also give an example of a connection in the normal bundle of a submanifold of a Riemannian manifold and study its properties.

On the  $m$ -dimensional real vector space  $\mathbb{R}^m$  we have the well-known differential operator

$$\partial : C^\infty(T\mathbb{R}^m) \times C^\infty(T\mathbb{R}^m) \rightarrow C^\infty(T\mathbb{R}^m)$$

mapping a pair of vector fields  $X, Y$  on  $\mathbb{R}^m$  to the **directional derivative**  $\partial_X Y$  of  $Y$  in the direction of  $X$  given by

$$(\partial_X Y)(x) = \lim_{t \rightarrow 0} \frac{Y(x + tX(x)) - Y(x)}{t}.$$

The most fundamental properties of the operator  $\partial$  are expressed by the following. If  $\lambda, \mu \in \mathbb{R}$ ,  $f, g \in C^\infty(\mathbb{R}^m)$  and  $X, Y, Z \in C^\infty(T\mathbb{R}^m)$  then

- (i)  $\partial_X(\lambda \cdot Y + \mu \cdot Z) = \lambda \cdot \partial_X Y + \mu \cdot \partial_X Z$ ,
- (ii)  $\partial_X(f \cdot Y) = X(f) \cdot Y + f \cdot \partial_X Y$ ,
- (iii)  $\partial(f \cdot X + g \cdot Y)Z = f \cdot \partial_X Z + g \cdot \partial_Y Z$ .

The next result shows that the differential operator  $\partial$  is compatible with both the standard differentiable structure on  $\mathbb{R}^m$  and its Euclidean metric.

**Proposition 6.1.** *Let the real vector space  $\mathbb{R}^m$  be equipped with the standard Euclidean metric  $\langle \cdot, \cdot \rangle$  and  $X, Y, Z \in C^\infty(T\mathbb{R}^m)$  be smooth vector fields on  $\mathbb{R}^m$ . Then*

- (iv)  $\partial_X Y - \partial_Y X = [X, Y]$ ,
- (v)  $X(\langle Y, Z \rangle) = \langle \partial_X Y, Z \rangle + \langle Y, \partial_X Z \rangle$ .

We shall now generalize the differential operator  $\partial$  on the Euclidean space  $\mathbb{R}^m$  to the so called Levi-Civita connection  $\nabla$  on a Riemannian manifold  $(M, g)$ . First we introduce the concept of a connection in a smooth vector bundle.

**Definition 6.2.** Let  $(E, M, \pi)$  be a smooth vector bundle over  $M$ . A **connection** on  $(E, M, \pi)$  is a map  $\hat{\nabla} : C^\infty(TM) \times C^\infty(E) \rightarrow C^\infty(E)$  such that

- (i)  $\hat{\nabla}_X(\lambda \cdot v + \mu \cdot w) = \lambda \cdot \hat{\nabla}_X v + \mu \cdot \hat{\nabla}_X w$ ,
- (ii)  $\hat{\nabla}_X(f \cdot v) = X(f) \cdot v + f \cdot \hat{\nabla}_X v$ ,
- (iii)  $\hat{\nabla}_{(f \cdot X + g \cdot Y)} v = f \cdot \hat{\nabla}_X v + g \cdot \hat{\nabla}_Y v$ .

for all  $\lambda, \mu \in \mathbb{R}$ ,  $X, Y \in C^\infty(TM)$ ,  $v, w \in C^\infty(E)$  and  $f, g \in C^\infty(M)$ . A section  $v \in C^\infty(E)$  of the vector bundle  $E$  is said to be **parallel** with respect to the connection  $\hat{\nabla}$  if

$$\hat{\nabla}_X v = 0$$

for all vector fields  $X \in C^\infty(TM)$ .

**Definition 6.3.** Let  $M$  be a smooth manifold and  $\hat{\nabla}$  be a connection on the tangent bundle  $(TM, M, \pi)$ . Then we define the **torsion**  $T : C_2^\infty(TM) \rightarrow C_1^\infty(TM)$  of  $\hat{\nabla}$  by

$$T(X, Y) = \hat{\nabla}_X Y - \hat{\nabla}_Y X - [X, Y],$$

where  $[,]$  is the Lie bracket on  $C^\infty(TM)$ . The connection  $\hat{\nabla}$  is said to be **torsion-free** if its torsion  $T$  vanishes i.e.

$$[X, Y] = \hat{\nabla}_X Y - \hat{\nabla}_Y X$$

for all  $X, Y \in C^\infty(TM)$ .

**Definition 6.4.** Let  $(M, g)$  be a Riemannian manifold. Then a connection  $\hat{\nabla}$  on the tangent bundle  $(TM, M, \pi)$  is said to be **metric** or compatible with the Riemannian metric  $g$  if

$$X(g(Y, Z)) = g(\hat{\nabla}_X Y, Z) + g(Y, \hat{\nabla}_X Z)$$

for all  $X, Y, Z \in C^\infty(TM)$ .

Let  $(M, g)$  be a Riemannian manifold and  $\nabla$  be a metric and torsion-free connection on its tangent bundle  $(TM, M, \pi)$ . Then it is easily seen that the following equations hold

$$g(\nabla_X Y, Z) = X(g(Y, Z)) - g(Y, \nabla_X Z),$$

$$g(\nabla_X Y, Z) = g([X, Y], Z) + g(\nabla_Y X, Z)$$

$$= g([X, Y], Z) + Y(g(X, Z)) - g(X, \nabla_Y Z),$$

$$\begin{aligned} 0 &= -Z(g(X, Y)) + g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \\ &= -Z(g(X, Y)) + g(\nabla_X Z + [Z, X], Y) + g(X, \nabla_Y Z - [Y, Z]). \end{aligned}$$

By adding these relations we yield

$$\begin{aligned} 2 \cdot g(\nabla_X Y, Z) &= \{X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) \\ &\quad + g(Z, [X, Y]) + g(Y, [Z, X]) - g(X, [Y, Z])\}. \end{aligned}$$

If  $\{E_1, \dots, E_m\}$  is a local orthonormal frame for the tangent bundle then

$$\nabla_X Y = \sum_{k=1}^m g(\nabla_X Y, E_k) E_k.$$

As a direct consequence there exists **at most one** metric and torsion-free connection on the tangent bundle.

**Definition 6.5.** Let  $(M, g)$  be a Riemannian manifold then the map  $\nabla : C^\infty(TM) \times C^\infty(TM) \rightarrow C^\infty(TM)$  given by

$$\begin{aligned} 2 \cdot g(\nabla_X Y, Z) &= \{X(g(Y, Z)) + Y(g(X, Z)) - Z(g(X, Y)) \\ &\quad + g([Z, X], Y) + g([Z, Y], X) + g(Z, [X, Y])\}. \end{aligned}$$

is called the **Levi-Civita connection** on  $M$ .

**Remark 6.6.** It is very important to note that the Levi-Civita connection is an intrinsic object on  $(M, g)$  i.e. only depending on the differentiable structure of the manifold and its Riemannian metric.

**Proposition 6.7.** *Let  $(M, g)$  be a Riemannian manifold. Then the Levi-Civita connection  $\nabla$  is a connection on the tangent bundle  $TM$  of  $M$ .*

**PROOF.** It follows from Definition 3.6, Theorem 4.16 and the fact that  $g$  is a tensor field that

$$g(\nabla_X(\lambda \cdot Y_1 + \mu \cdot Y_2), Z) = \lambda \cdot g(\nabla_X Y_1, Z) + \mu \cdot g(\nabla_X Y_2, Z)$$

and

$$g(\nabla_{Y_1 + Y_2} X, Z) = g(\nabla_{Y_1} X, Z) + g(\nabla_{Y_2} X, Z)$$

for all  $\lambda, \mu \in \mathbb{R}$  and  $X, Y_1, Y_2, Z \in C^\infty(TM)$ . Furthermore we have for all  $f \in C^\infty(M)$

$$\begin{aligned} &2 \cdot g(\nabla_X f Y, Z) \\ &= \{X(f \cdot g(Y, Z)) + f \cdot Y(g(X, Z)) - Z(f \cdot g(X, Y)) \\ &\quad + f \cdot g([Z, X], Y) + g([Z, f \cdot Y], X) + g(Z, [X, f \cdot Y])\} \end{aligned}$$

$$\begin{aligned}
&= \{X(f) \cdot g(Y, Z) + f \cdot X(g(Y, Z)) + f \cdot Y(g(X, Z)) \\
&\quad - Z(f) \cdot g(X, Y) - f \cdot Z(g(X, Y)) + f \cdot g([Z, X], Y) \\
&\quad + g(Z(f) \cdot Y + f \cdot [Z, Y], X) + g(Z, X(f) \cdot Y + f \cdot [X, Y])\} \\
&= 2 \cdot \{X(f) \cdot g(Y, Z) + f \cdot g(\nabla_X Y, Z)\} \\
&= 2 \cdot g(X(f) \cdot Y + f \cdot \nabla_X Y, Z)
\end{aligned}$$

and

$$\begin{aligned}
&2 \cdot g(\nabla_f \cdot X Y, Z) \\
&= \{f \cdot X(g(Y, Z)) + Y(f \cdot g(X, Z)) - Z(f \cdot g(X, Y)) \\
&\quad + g([Z, f \cdot X], Y) + f \cdot g([Z, Y], X) + g(Z, [f \cdot X, Y])\} \\
&= \{f \cdot X(g(Y, Z)) + Y(f) \cdot g(X, Z) + f \cdot Y(g(X, Z)) \\
&\quad - Z(f) \cdot g(X, Y) - f \cdot Z(g(X, Y)) \\
&\quad + g(Z(f) \cdot X, Y) + f \cdot g([Z, X], Y) \\
&\quad + f \cdot g([Z, Y], X) + f \cdot g(Z, [X, Y]) - g(Z, Y(f) \cdot X)\} \\
&= 2 \cdot f \cdot g(\nabla_X Y, Z).
\end{aligned}$$

This proves that  $\nabla$  is a connection on the tangent bundle  $(TM, M, \pi)$ .  $\square$

The next result is called the Fundamental Theorem of Riemannian geometry.

**Theorem 6.8.** *Let  $(M, g)$  be a Riemannian manifold. Then the Levi-Civita connection is the unique metric and torsion-free connection on the tangent bundle  $(TM, M, \pi)$ .*

PROOF. The difference  $g(\nabla_X Y, Z) - g(\nabla_Y X, Z)$  equals twice the skew-symmetric part (w.r.t the pair  $(X, Y)$ ) of the right hand side of the equation in Definition 6.5. This is the same as

$$= \frac{1}{2} \{g(Z, [X, Y]) - g(Z, [Y, X])\} = g(Z, [X, Y]).$$

This proves that the Levi-Civita connection is torsion-free.

The sum  $g(\nabla_X Y, Z) + g(\nabla_X Z, Y)$  equals twice the symmetric part (w.r.t the pair  $(Y, Z)$ ) on the right hand side of Definition 6.5. This is exactly

$$= \frac{1}{2} \{X(g(Y, Z)) + X(g(Z, Y))\} = X(g(Y, Z)).$$

This shows that the Levi-Civita connection is compatible with the Riemannian metric  $g$  on  $M$ .  $\square$

A vector field  $X \in C^\infty(TM)$  on  $(M, g)$  induces the **first order covariant derivative**

$$\nabla_X : C^\infty(TM) \rightarrow C^\infty(TM)$$

in the direction of  $X$  by

$$\nabla_X : Y \mapsto \nabla_X Y.$$

**Definition 6.9.** Let  $G$  be a Lie group. For a left-invariant vector field  $Z \in \mathfrak{g}$  we define the map  $\text{ad}_Z : \mathfrak{g} \rightarrow \mathfrak{g}$  by

$$\text{ad}_Z : X \mapsto [Z, X].$$

**Proposition 6.10.** Let  $(G, g)$  be a Lie group equipped with a left-invariant metric. Then the Levi-Civita connection  $\nabla$  satisfies

$$g(\nabla_X Y, Z) = \frac{1}{2} \{g([X, Y], Z) + g(\text{ad}_Z(X), Y) + g(X, \text{ad}_Z(Y))\}$$

for all  $X, Y, Z \in \mathfrak{g}$ . In particular, if for all  $Z \in \mathfrak{g}$  the map  $\text{ad}_Z$  is skew symmetric with respect to  $g$  then

$$\nabla_X Y = \frac{1}{2}[X, Y].$$

PROOF. See Exercise 6.2. □

**Proposition 6.11.** Let  $G$  be one of the classical compact Lie groups  $\mathbf{O}(m)$ ,  $\mathbf{SO}(m)$ ,  $\mathbf{U}(m)$  or  $\mathbf{SU}(m)$  equipped with the left-invariant metric

$$g(Z, W) = \text{Re}(\text{trace}(\bar{Z}^t W)).$$

Then for each  $X \in \mathfrak{g}$  the operator  $\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$  is skew symmetric.

PROOF. See Exercise 6.3. □

**Example 6.12.** Let  $(M, g)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$ . Further let  $(U, x)$  be local coordinates on  $M$  and put  $X_i = \partial/\partial x_i \in C^\infty(TU)$ . Then  $\{X_1, \dots, X_m\}$  is a local frame of  $TM$  on  $U$ . For  $(U, x)$  we define the **Christoffel symbols**  $\Gamma_{ij}^k : U \rightarrow \mathbb{R}$  of the connection  $\nabla$  with respect to  $(U, x)$  by

$$\sum_{k=1}^m \Gamma_{ij}^k X_k = \nabla_{X_i} X_j.$$

On the subset  $x(U)$  of  $\mathbb{R}^m$  we define the metric  $\tilde{g}$  by

$$\tilde{g}(e_i, e_j) = g_{ij} = g(X_i, X_j).$$

The differential  $dx$  is bijective so Proposition 4.21 implies that

$$dx([X_i, X_j]) = [dx(X_i), dx(X_j)] = [\partial_{e_i}, \partial_{e_j}] = 0$$

and hence  $[X_i, X_j] = 0$ . From the definition of the Levi-Civita connection we now get

$$\begin{aligned} \sum_{k=1}^m \Gamma_{ij}^k g_{kl} &= \left\langle \sum_{k=1}^m \Gamma_{ij}^k X_k, X_l \right\rangle \\ &= \langle \nabla_{X_i} X_j, X_l \rangle \\ &= \frac{1}{2} \{X_i \langle X_j, X_l \rangle + X_j \langle X_l, X_i \rangle - X_l \langle X_i, X_j \rangle\} \\ &= \frac{1}{2} \left\{ \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right\}. \end{aligned}$$

If  $g^{kl} = (g^{-1})_{kl}$  then

$$\Gamma_{ij}^k = \frac{1}{2} \sum_{l=1}^m g^{kl} \left\{ \frac{\partial g_{jl}}{\partial x_i} + \frac{\partial g_{li}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \right\}.$$

**Definition 6.13.** Let  $N$  be a smooth manifold,  $M$  be a submanifold of  $N$  and  $\tilde{X} \in C^\infty(TM)$  be a vector field on  $M$ . Let  $U$  be an open subset of  $N$  such that  $U \cap M \neq \emptyset$ . A **local extension** of  $\tilde{X}$  to  $U$  is a vector field  $X \in C^\infty(TU)$  such that  $\tilde{X}_p = X_p$  for all  $p \in U \cap M$ . If  $U = N$  then  $X$  is called a **global extension**.

**Fact 6.14.** Let  $M$  be a submanifold of a Riemannian manifold  $(N, h)$ ,  $\tilde{X} \in C^\infty(TM)$ ,  $\tilde{Y} \in C^\infty(NM)$  and  $p \in M$ . Then there exists an open neighbourhood  $U$  of  $N$  containing  $p$  and  $X, Y \in C^\infty(TU)$  extending  $\tilde{X}$  and  $\tilde{Y}$ , respectively, on  $U$ .

Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold equipped with the induced metric  $g$ . Let  $Z \in C^\infty(TN)$  be a vector field on  $N$  and  $\tilde{Z} = Z|_M : M \rightarrow TN$  be the restriction of  $Z$  to  $M$ . Note that  $\tilde{Z}$  is not necessarily an element of  $C^\infty(TM)$  i.e. a vector field on the submanifold  $M$ . For each  $p \in M$  the tangent vector  $\tilde{Z}_p \in T_p N$  can be decomposed

$$\tilde{Z}_p = \tilde{Z}_p^\top + \tilde{Z}_p^\perp$$

in a unique way into its tangential part  $(\tilde{Z}_p)^\top \in T_p M$  and its normal part  $(\tilde{Z}_p)^\perp \in N_p M$ . For this we write  $\tilde{Z} = \tilde{Z}^\top + \tilde{Z}^\perp$ .

Let  $\tilde{X}, \tilde{Y} \in C^\infty(TM)$  be vector fields on  $M$  extended by  $X, Y \in C^\infty(TN)$  to  $N$ . If  $p \in M$  then  $(\nabla_X Y)_p$  only depends on the value  $X_p = \tilde{X}_p$  and the value of  $Y$  along some curve  $\gamma : (-\epsilon, \epsilon) \rightarrow N$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = X_p = \tilde{X}_p$ . For this see Remark 7.3. Since  $X_p \in T_p M$  we may choose the curve  $\gamma$  such that the image  $\gamma((-\epsilon, \epsilon))$  is contained in  $M$ . Then  $\tilde{Y}_{\gamma(t)} = Y_{\gamma(t)}$  for  $t \in (-\epsilon, \epsilon)$ . This means



that  $(\nabla_X Y)_p$  only depends on  $\tilde{X}_p$  and the value of  $\tilde{Y}$  along  $\gamma$ , hence independent of the way  $\tilde{X}$  and  $\tilde{Y}$  are extended. This shows that the following operators  $\tilde{\nabla}$  and  $B$  are well defined.

**Definition 6.15.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold with the induced metric  $g$ . Then we define

$$\tilde{\nabla} : C^\infty(TM) \times C^\infty(TM) \rightarrow C^\infty(TM)$$

and

$$B : C^\infty(TM) \times C^\infty(TM) \rightarrow C^\infty(NM)$$

by

$$\tilde{\nabla}_{\tilde{X}} \tilde{Y} = (\nabla_X Y)^\top \quad \text{and} \quad B(\tilde{X}, \tilde{Y}) = (\nabla_X Y)^\perp,$$

where  $X, Y \in C^\infty(TN)$  are any extensions of  $\tilde{X}, \tilde{Y}$ . The operator  $B$  is called the **second fundamental form** of  $M$  in  $(N, h)$ .

**Theorem 6.16.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold with the induced metric  $g$ . Then the operator  $\tilde{\nabla}$  is the Levi-Civita connection of the submanifold  $(M, g)$ .

PROOF. See Exercise 6.8. □

**Proposition 6.17.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold with the induced metric  $g$ . Then the second fundamental form  $B$  of  $M$  in  $(N, h)$  is symmetric and tensorial in both its arguments.

PROOF. See Exercise 6.7. □

**Definition 6.18.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold with the induced metric  $g$ . Then  $M$  is said to be **minimal** if its second fundamental form

$$B : C^\infty(TM) \otimes C^\infty(TM) \rightarrow C^\infty(NM)$$

is traceless i.e.

$$\text{trace} B = \sum_{k=1}^m B(X_k, X_k) = 0.$$

Here  $\{X_1, X_2, \dots, X_m\}$  is any local orthonormal frame for the tangent bundle  $TM$ .

Let us now consider the classical situation of a regular surface  $\Sigma$  as a submanifold of the 3-dimensional Euclidean space  $\mathbb{R}^3$ . Let  $\{\tilde{X}, \tilde{Y}\}$  be a local orthonormal frame for the tangent bundle  $T\Sigma$  of  $\Sigma$  around a point  $p \in \Sigma$  and  $\tilde{N}$  be the local Gauss map with  $\tilde{N} = \tilde{X} \times \tilde{Y}$ . If

$X, Y, N$  are local extensions of  $\tilde{X}, \tilde{Y}, \tilde{N}$  then the second fundamental form  $B$  of  $\Sigma$  in  $\mathbb{R}^3$  satisfies

$$\begin{aligned} B(\tilde{X}, \tilde{Y}) &= \langle \partial_X Y, N \rangle N \\ &= - \langle Y, \partial_X N \rangle N \\ &= - \langle Y, dN(X) \rangle N \\ &= \langle Y, S_p(X) \rangle N, \end{aligned}$$

where  $S_p : T_p \Sigma \rightarrow T_p \Sigma$  is the shape operator at  $p$ . The trace of  $B$  then satisfies

$$\text{trace} B = (\langle S_p(X), X \rangle + \langle S_p(Y), Y \rangle) N = (k_1 + k_2) N = 0$$

if and only if the classical **mean curvature**  $H = (k_1 + k_2)/2$  vanishes.

We conclude this chapter by observing that the Levi-Civita connection of a Riemannian  $(N, h)$  induces a metric connection  $\bar{\nabla}$  on the normal bundle  $NM$  of its submanifold  $M$  as follows.

**Proposition 6.19.** *Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold with the induced metric  $g$ . Let  $X, Y \in C^\infty(TM)$  be vector fields extending  $\tilde{X} \in C^\infty(TM)$  and  $\tilde{Y} \in C^\infty(NM)$ . Then the map  $\bar{\nabla} : C^\infty(TM) \times C^\infty(NM) \rightarrow C^\infty(NM)$  given by*

$$\bar{\nabla}_{\tilde{X}} \tilde{Y} = (\nabla_X Y)^\perp$$

is a well-defined connection on the normal bundle  $NM$  satisfying

$$\tilde{X}(h(\tilde{Y}, \tilde{Z})) = h(\bar{\nabla}_{\tilde{X}} \tilde{Y}, \tilde{Z}) + h(\tilde{Y}, \bar{\nabla}_{\tilde{X}} \tilde{Z})$$

for all  $\tilde{X} \in C^\infty(TM)$  and  $\tilde{Y}, \tilde{Z} \in C^\infty(NM)$ .

PROOF. See Exercise 6.9. □

## Exercises

**Exercise 6.1.** Let  $M$  be a smooth manifold and  $\hat{\nabla}$  be a connection on the tangent bundle  $(TM, M, \pi)$ . Prove that the torsion  $T : C_2^\infty(TM) \rightarrow C_1^\infty(TM)$  of  $\hat{\nabla}$  is a tensor field of type  $(2, 1)$ .

**Exercise 6.2.** Find a proof of Proposition 6.10.

**Exercise 6.3.** Find a proof of Proposition 6.11.

**Exercise 6.4.** Let  $\text{Sol}^3$  be the 3-dimensional subgroup of  $\mathbf{SL}_3(\mathbb{R})$  given by

$$\text{Sol}^3 = \left\{ \begin{pmatrix} e^z & 0 & x \\ 0 & e^{-z} & y \\ 0 & 0 & 1 \end{pmatrix} \mid p = (x, y, z) \in \mathbb{R}^3 \right\}.$$

Let  $X, Y, Z \in \mathfrak{g}$  be left-invariant vector fields on  $\text{Sol}^3$  such that

$$X_e = \frac{\partial}{\partial x} \Big|_{p=0}, \quad Y_e = \frac{\partial}{\partial y} \Big|_{p=0} \quad \text{and} \quad Z_e = \frac{\partial}{\partial z} \Big|_{p=0}.$$

Show that

$$[X, Y] = 0, \quad [Z, X] = X \quad \text{and} \quad [Z, Y] = -Y.$$

Let  $g$  be a left-invariant Riemannian metric on  $G$  such that  $\{X, Y, Z\}$  is an orthonormal basis for the Lie algebra  $\mathfrak{g}$ . Calculate the vector fields

$$\nabla_X Y, \quad \nabla_Y X, \quad \nabla_X Z, \quad \nabla_Z X, \quad \nabla_Y Z \quad \text{and} \quad \nabla_Z Y.$$

**Exercise 6.5.** Let  $\mathbf{SO}(m)$  be the special orthogonal group equipped with the metric

$$\langle X, Y \rangle = \frac{1}{2} \text{trace}(X^t Y).$$

Prove that  $\langle \cdot, \cdot \rangle$  is left-invariant and that for left-invariant vector fields  $X, Y \in \mathfrak{so}(m)$  we have

$$\nabla_X Y = \frac{1}{2} [X, Y].$$

Let  $A, B, C$  be elements of the Lie algebra  $\mathfrak{so}(3)$  with

$$A_e = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_e = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad C_e = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Prove that  $\{A, B, C\}$  is an orthonormal basis for  $\mathfrak{so}(3)$  and calculate

$$\nabla_A B, \quad \nabla_B C \quad \text{and} \quad \nabla_C A.$$

**Exercise 6.6.** Let  $\mathbf{SL}_2(\mathbb{R})$  be the real special linear group equipped with the metric

$$\langle X, Y \rangle_p = \frac{1}{2} \operatorname{trace}((p^{-1}X)^t(p^{-1}Y)).$$

Let  $A, B, C$  be elements of the Lie algebra  $\mathfrak{sl}_2(\mathbb{R})$  with

$$A_e = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad B_e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad C_e = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Prove that  $\{A, B, C\}$  is an orthonormal basis for  $\mathfrak{sl}_2(\mathbb{R})$  and calculate

$$\nabla_A B, \quad \nabla_B C \quad \text{and} \quad \nabla_C A.$$

**Exercise 6.7.** Find a proof of Proposition 6.17.

**Exercise 6.8.** Find a proof of Theorem 6.16.

**Exercise 6.9.** Find a proof of Proposition 6.19.

## Geodesics

In this chapter we introduce the notion of a geodesic on a Riemannian manifold  $(M, g)$ . This is a solution to a second order non-linear system of ordinary differential equations. We show that geodesics are solutions to two different variational problems. They are critical points to the so called energy functional and furthermore locally shortest paths between their endpoints.

**Definition 7.1.** Let  $M$  be a smooth manifold and  $(TM, M, \pi)$  be its tangent bundle. A **vector field  $X$  along a curve  $\gamma : I \rightarrow M$**  is a curve  $X : I \rightarrow TM$  such that  $\pi \circ X = \gamma$ . By  $C_\gamma^\infty(TM)$  we denote the set of all smooth vector fields along  $\gamma$ . For  $X, Y \in C_\gamma^\infty(TM)$  and  $f \in C^\infty(I)$  we define the operations  $+$  and  $\cdot$  by

- (i)  $(X + Y)(t) = X(t) + Y(t)$ ,
- (ii)  $(f \cdot X)(t) = f(t) \cdot X(t)$ .

This turns  $(C_\gamma^\infty(TM), +, \cdot)$  into a module over  $C^\infty(I)$  and a real vector space over the constant functions in particular. For a given smooth curve  $\gamma : I \rightarrow M$  in  $M$  the smooth vector field  $X : I \rightarrow TM$  with  $X : t \mapsto (\gamma(t), \dot{\gamma}(t))$  is called the **tangent field** along  $\gamma$ .

The next result gives a rule for differentiating a vector field along a given curve and shows how this is related to the Levi-Civita connection.

**Proposition 7.2.** *Let  $(M, g)$  be a smooth Riemannian manifold and  $\gamma : I \rightarrow M$  be a curve in  $M$ . Then there exists a unique operator*

$$\frac{D}{dt} : C_\gamma^\infty(TM) \rightarrow C_\gamma^\infty(TM)$$

such that for all  $\lambda, \mu \in \mathbb{R}$  and  $f \in C^\infty(I)$ ,

- (i)  $D(\lambda \cdot X + \mu \cdot Y)/dt = \lambda \cdot (DX/dt) + \mu \cdot (DY/dt)$ ,
- (ii)  $D(f \cdot Y)/dt = df/dt \cdot Y + f \cdot (DY/dt)$ , and
- (iii) for each  $t_0 \in I$  there exists an open subinterval  $J_0$  of  $I$  such that  $t_0 \in J_0$  and if  $X \in C^\infty(TM)$  is a vector field with  $X_{\gamma(t)} = Y(t)$  for all  $t \in J_0$  then

$$\left(\frac{DY}{dt}\right)(t_0) = (\nabla_{\dot{\gamma}} X)_{\gamma(t_0)}.$$

PROOF. Let us first prove the uniqueness, so for the moment we assume that such an operator exists. For a point  $t_0 \in I$  choose a chart  $(U, x)$  on  $M$  and open subinterval  $J \subset I$  such that  $t_0 \in J$ ,  $\gamma(J) \subset U$  and put  $X_i = \partial/\partial x_i \in C^\infty(TU)$ . Then any vector field  $Y$  along the restriction of  $\gamma$  to  $J$  can be written in the form

$$Y(t) = \sum_{j=1}^m \alpha_j(t) (X_j)_{\gamma(t)}$$

for some functions  $\alpha_j \in C^\infty(J)$ . The second condition means that

$$(1) \quad \left(\frac{DY}{dt}\right)(t) = \sum_{j=1}^m \alpha_j(t) \left(\frac{DX_j}{dt}\right)_{\gamma(t)} + \sum_{k=1}^m \dot{\alpha}_k(t) (X_k)_{\gamma(t)}.$$

Let  $x \circ \gamma(t) = (\gamma_1(t), \dots, \gamma_m(t))$  then

$$\dot{\gamma}(t) = \sum_{i=1}^m \dot{\gamma}_i(t) (X_i)_{\gamma(t)}$$

and the third condition for  $D/dt$  implies that

$$(2) \quad \left(\frac{DX_j}{dt}\right)_{\gamma(t)} = (\nabla_{\dot{\gamma}} X_j)_{\gamma(t)} = \sum_{i=1}^m \dot{\gamma}_i(t) (\nabla_{X_i} X_j)_{\gamma(t)}.$$

Together equations (1) and (2) give

$$(3) \quad \left(\frac{DY}{dt}\right)(t) = \sum_{k=1}^m \left( \dot{\alpha}_k(t) + \sum_{i,j=1}^m \Gamma_{ij}^k(\gamma(t)) \dot{\gamma}_i(t) \alpha_j(t) \right) (X_k)_{\gamma(t)}.$$

This shows that the operator  $D/dt$  is uniquely determined.

It is easily seen that if we use equation (3) for defining an operator  $D/dt$  then it satisfies the necessary conditions of Proposition 7.2. This proves the existence of the operator  $D/dt$ .  $\square$

**Remark 7.3.** It follows from the fact that the Levi-Civita connection is tensorial in its first argument i.e.

$$\nabla_f \cdot Z X = f \cdot \nabla_Z X$$

and the equation

$$(\nabla_{\dot{\gamma}} X)_{\gamma(t_0)} = \left(\frac{DY}{dt}\right)(t_0)$$

in Proposition 7.2 that the value  $(\nabla_Z X)_p$  of  $\nabla_Z X$  at  $p$  only depends on the value of  $Z_p$  of  $Z$  at  $p$  and the values of  $Y$  along some curve  $\gamma$  satisfying  $\gamma(0) = p$  and  $\dot{\gamma}(0) = Z_p$ . This allows us to use the notation  $\nabla_{\dot{\gamma}} Y$  for  $DY/dt$ .

The Levi-Civita connection can now be used to define the notions of parallel vector fields and geodesics on Riemannian manifolds. We will show that they are solutions to ordinary differential equations.

**Definition 7.4.** Let  $(M, g)$  be a Riemannian manifold and  $\gamma : I \rightarrow M$  be a  $C^1$ -curve. A vector field  $X$  along  $\gamma$  is said to be **parallel** if

$$\nabla_{\dot{\gamma}} X = 0.$$

A  $C^2$ -curve  $\gamma : I \rightarrow M$  on  $M$  is said to be a **geodesic** if its tangent field  $\dot{\gamma}$  is parallel along  $\gamma$  i.e.

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0.$$

The next result shows that for given initial values at a point  $p \in M$  we get a parallel vector field globally defined along any curve through that point.

**Theorem 7.5.** *Let  $(M, g)$  be a Riemannian manifold and  $I = (a, b)$  be an open interval on the real line  $\mathbb{R}$ . Further let  $\gamma : [a, b] \rightarrow M$  be a continuous curve which is  $C^1$  on  $I$ ,  $t_0 \in I$  and  $X_0 \in T_{\gamma(t_0)}M$ . Then there exists a unique parallel vector field  $Y$  along  $\gamma$  such that  $X_0 = Y(t_0)$ .*

**PROOF.** Let  $(U, x)$  be a chart on  $M$  such that  $\gamma(t_0) \in U$  and put  $X_i = \partial/\partial x_i \in C^\infty(TU)$ . Let  $J$  be an open subset of  $I$  such that the image  $\gamma(J)$  is contained in  $U$ . Then the tangent of the restriction of  $\gamma$  to  $J$  can be written as

$$\dot{\gamma}(t) = \sum_{i=1}^m \dot{\gamma}_i(t) (X_i)_{\gamma(t)}.$$

Similarly, let  $Y$  be a vector field along  $\gamma$  represented by

$$Y(t) = \sum_{j=1}^m \alpha_j(t) (X_j)_{\gamma(t)}.$$

Then

$$\begin{aligned} (\nabla_{\dot{\gamma}} Y)(t) &= \sum_{j=1}^m \{ \dot{\alpha}_j(t) (X_j)_{\gamma(t)} + \alpha_j(t) (\nabla_{\dot{\gamma}} X_j)_{\gamma(t)} \} \\ &= \sum_{k=1}^m \{ \dot{\alpha}_k(t) + \sum_{i,j=1}^m \alpha_j(t) \dot{\gamma}_i(t) \Gamma_{ij}^k(\gamma(t)) \} (X_k)_{\gamma(t)}. \end{aligned}$$

This implies that the vector field  $Y$  is parallel i.e.  $\nabla_{\dot{\gamma}} Y \equiv 0$  if and only if the following first order **linear** system of ordinary differential

equations is satisfied

$$\dot{\alpha}_k(t) + \sum_{i,j=1}^m \alpha_j(t) \dot{\gamma}_i(t) \Gamma_{ij}^k(\gamma(t)) = 0$$

for all  $k = 1, \dots, m$ . It follows from Fact 7.6 that to each initial value  $\alpha(t_0) = (v_1, \dots, v_m) \in \mathbb{R}^m$  with

$$Y_0 = \sum_{k=1}^m v_k (X_k)_{\gamma(t_0)}$$

there exists a unique solution  $\alpha = (\alpha_1, \dots, \alpha_m)$  to the above system. This gives us the unique parallel vector field  $Y$

$$Y(t) = \sum_{k=1}^m \alpha_k(t) (X_k)_{\gamma(t)}$$

along  $J$ . Since the Christoffel symbols are bounded along the compact set  $[a, b]$  it is clear that the parallel vector field can be extended to the whole of  $I = (a, b)$ .  $\square$

The following result is the well-known theorem of Picard-Lindelöf.

**Fact 7.6.** *Let  $f : U \rightarrow \mathbb{R}^n$  be a continuous map defined on an open subset  $U$  of  $\mathbb{R} \times \mathbb{R}^n$  and  $L \in \mathbb{R}^+$  such that*

$$|f(t, y_1) - f(t, y_2)| \leq L \cdot |y_1 - y_2|$$

*for all  $(t, y_1), (t, y_2) \in U$ . If  $(t_0, x_0) \in U$  then there exists a unique local solution  $x : I \rightarrow \mathbb{R}^n$  to the following initial value problem*

$$x'(t) = f(t, x(t)), \quad x(t_0) = x_0.$$

**Lemma 7.7.** *Let  $(M, g)$  be a Riemannian manifold,  $\gamma : I \rightarrow M$  be a smooth curve and  $X, Y$  be parallel vector fields along  $\gamma$ . Then the function  $g(X, Y) : I \rightarrow \mathbb{R}$  given by  $t \mapsto g_{\gamma(t)}(X_{\gamma(t)}, Y_{\gamma(t)})$  is constant. In particular, if  $\gamma$  is a geodesic then  $g(\dot{\gamma}, \dot{\gamma})$  is constant along  $\gamma$ .*

**PROOF.** Using the fact that the Levi-Civita connection is metric we obtain

$$\frac{d}{dt}(g(X, Y)) = g(\nabla_{\dot{\gamma}} X, Y) + g(X, \nabla_{\dot{\gamma}} Y) = 0.$$

This proves that the function  $g(X, Y)$  is constant along  $\gamma$ .  $\square$

The following result on parallel vector fields is a useful tool in Riemannian geometry. It turns out to be very useful in Chapter 9.



**Proposition 7.8.** *Let  $(M, g)$  be a Riemannian manifold,  $p \in M$  and  $\{v_1, \dots, v_m\}$  be an orthonormal basis for the tangent space  $T_p M$ . Let  $\gamma : I \rightarrow M$  be a smooth curve such that  $\gamma(0) = p$  and  $X_1, \dots, X_m$  be parallel vector fields along  $\gamma$  such that  $X_k(0) = v_k$  for  $k = 1, 2, \dots, m$ . Then the set  $\{X_1(t), \dots, X_m(t)\}$  is a orthonormal basis for the tangent space  $T_{\gamma(t)} M$  for all  $t \in I$ .*

PROOF. This is a direct consequence of Lemma 7.7.  $\square$

Geodesics are of great importance in Riemannian geometry. For those we have the following fundamental uniqueness and existence result.

**Theorem 7.9.** *Let  $(M, g)$  be a Riemannian manifold. If  $p \in M$  and  $v \in T_p M$  then there exists an open interval  $I = (-\epsilon, \epsilon)$  and a unique geodesic  $\gamma : I \rightarrow M$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ .*

PROOF. Let  $(U, x)$  be a chart on  $M$  such that  $p \in U$  and put  $X_i = \partial/\partial x_i \in C^\infty(TU)$ . Let  $J$  be an open subset of  $I$  such that the image  $\gamma(J)$  is contained in  $U$ . Then the tangent of the restriction of  $\gamma$  to  $J$  can be written as

$$\dot{\gamma}(t) = \sum_{i=1}^m \dot{\gamma}_i(t) (X_i)_{\gamma(t)}.$$

By differentiation we then obtain

$$\begin{aligned} \nabla_{\dot{\gamma}} \dot{\gamma} &= \sum_{j=1}^m \nabla_{\dot{\gamma}} (\dot{\gamma}_j(t) (X_j)_{\gamma(t)}) \\ &= \sum_{j=1}^m \{ \ddot{\gamma}_j(t) (X_j)_{\gamma(t)} + \sum_{i=1}^m \dot{\gamma}_i(t) \dot{\gamma}_j(t) (\nabla_{X_i} X_j)_{\gamma(t)} \} \\ &= \sum_{k=1}^m \{ \ddot{\gamma}_k(t) + \sum_{i,j=1}^m \dot{\gamma}_i(t) \dot{\gamma}_j(t) \Gamma_{ij}^k(\gamma(t)) \} (X_k)_{\gamma(t)}. \end{aligned}$$

Hence the curve  $\gamma$  is a geodesic if and only if

$$\ddot{\gamma}_k(t) + \sum_{i,j=1}^m \dot{\gamma}_i(t) \dot{\gamma}_j(t) \Gamma_{ij}^k(\gamma(t)) = 0$$

for all  $k = 1, \dots, m$ . It follows from Fact 7.10 that for initial values  $q = x(p)$  and  $w = (dx)_p(v)$  there exists an open interval  $(-\epsilon, \epsilon)$  and a unique solution  $(\gamma_1, \dots, \gamma_m)$  satisfying the initial conditions

$$(\gamma_1(0), \dots, \gamma_m(0)) = q \quad \text{and} \quad (\dot{\gamma}_1(0), \dots, \dot{\gamma}_m(0)) = w.$$

$\square$

The following result is a second order consequence of the well-known theorem of Picard-Lindelöf.

**Fact 7.10.** *Let  $f : U \rightarrow \mathbb{R}^n$  be a continuous map defined on an open subset  $U$  of  $\mathbb{R} \times \mathbb{R}^{2n}$  and  $L \in \mathbb{R}^+$  such that*

$$|f(t, y_1) - f(t, y_2)| \leq L \cdot |y_1 - y_2|$$

*for all  $(t, y_1), (t, y_2) \in U$ . If  $(t_0, (x_0, x_1)) \in U$  and  $x_0, x_1 \in \mathbb{R}^n$  then there exists a unique local solution  $x : I \rightarrow \mathbb{R}^n$  to the following initial value problem*

$$x''(t) = f(t, x(t), x'(t)), \quad x(t_0) = x_0, \quad x'(t_0) = x_1.$$

**Remark 7.11.** The Levi-Civita connection  $\nabla$  on a given Riemannian manifold  $(M, g)$  is an inner object i.e. completely determined by the differentiable structure on  $M$  and the Riemannian metric  $g$ , see Remark 6.6. Hence the same applies for the condition

$$\nabla_{\dot{\gamma}} \dot{\gamma} = 0$$

for a given curve  $\gamma : I \rightarrow M$ . This means that the image of a geodesic under a local isometry is again a geodesic.

**Example 7.12.** Let  $E^m = (\mathbb{R}^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^m})$  be the Euclidean space. For the trivial chart  $\text{id}_{\mathbb{R}^m} : \mathbb{R}^m \rightarrow \mathbb{R}^m$  the metric on  $E^m$  is given by  $g_{ij} = \delta_{ij}$ . As a direct consequence of Example 6.12 we see that

$$\Gamma_{ij}^k = 0 \quad \text{for all } i, j, k = 1, \dots, m.$$

Hence  $\gamma : I \rightarrow \mathbb{R}^m$  is a geodesic if and only if  $\ddot{\gamma}(t) = 0$ . For  $p \in \mathbb{R}^m$  and  $v \in T_p \mathbb{R}^m \cong \mathbb{R}^m$  define

$$\gamma_{(p,v)} : \mathbb{R} \rightarrow \mathbb{R}^m \quad \text{by} \quad \gamma_{(p,v)}(t) = p + t \cdot v.$$

Then  $\gamma_{(p,v)}(0) = p$ ,  $\dot{\gamma}_{(p,v)}(0) = v$  and  $\ddot{\gamma}_{(p,v)} = 0$ . It now follows from Theorem 7.9 that the geodesics in  $E^m$  are the straight lines.

**Definition 7.13.** A geodesic  $\gamma : I \rightarrow (M, g)$  in a Riemannian manifold is said to be **maximal** if it cannot be extended to a geodesic defined on an interval  $J$  strictly containing  $I$ . The manifold  $(M, g)$  is said to be **complete** if for each point  $(p, v) \in TM$  there exists a geodesic  $\gamma : \mathbb{R} \rightarrow M$  defined on the whole of  $\mathbb{R}$  such that  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ .

**Proposition 7.14.** *Let  $(N, h)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$  and  $M$  be a submanifold equipped with the induced metric  $g$ . A curve  $\gamma : I \rightarrow M$  is a geodesic in  $M$  if and only if*

$$(\nabla_{\dot{\gamma}} \dot{\gamma})^\top = 0.$$

PROOF. Following Theorem 6.16 the Levi-Civita connection  $\tilde{\nabla}$  on  $(M, g)$  satisfies

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = (\nabla_{\dot{\gamma}}\dot{\gamma})^{\top}.$$

□

**Example 7.15.** Let  $E^{m+1}$  be the  $(m+1)$ -dimensional Euclidean space and  $S^m$  be the unit sphere in  $E^{m+1}$  with the induced metric. At a point  $p \in S^m$  the normal space  $N_p S^m$  of  $S^m$  in  $E^{m+1}$  is simply the line generated by  $p$ . If  $\gamma : I \rightarrow S^m$  is a curve on the sphere, then

$$\tilde{\nabla}_{\dot{\gamma}}\dot{\gamma} = (\nabla_{\dot{\gamma}}\dot{\gamma})^{\top} = (\partial_{\dot{\gamma}}\dot{\gamma})^{\top} = \ddot{\gamma}^{\top} = \ddot{\gamma} - \ddot{\gamma}^{\perp} = \ddot{\gamma} - \langle \ddot{\gamma}, \gamma \rangle \gamma.$$

This shows that  $\gamma$  is a geodesic on the sphere  $S^m$  if and only if

$$(4) \quad \ddot{\gamma} = \langle \ddot{\gamma}, \gamma \rangle \gamma.$$

For a point  $(p, X) \in TS^m$  define the curve  $\gamma = \gamma_{(p, X)} : \mathbb{R} \rightarrow S^m$  by

$$\gamma : t \mapsto \begin{cases} p & \text{if } X = 0 \\ \cos(|X|t) \cdot p + \sin(|X|t) \cdot X/|X| & \text{if } X \neq 0. \end{cases}$$

Then one easily checks that  $\gamma(0) = p$ ,  $\dot{\gamma}(0) = X$  and that  $\gamma$  satisfies the geodesic equation (4). This shows that the non-constant geodesics on  $S^m$  are precisely the great circles and the sphere is complete.

**Example 7.16.** Let  $\text{Sym}(\mathbb{R}^{m+1})$  be equipped with the metric

$$\langle A, B \rangle = \frac{1}{8} \text{trace}(A^t B).$$

Then we know that the map  $\phi : S^m \rightarrow \text{Sym}(\mathbb{R}^{m+1})$  with

$$\phi : p \mapsto (2pp^t - e)$$

is an isometric immersion and that the image  $\phi(S^m)$  is isometric to the  $m$ -dimensional real projective space  $\mathbb{R}P^m$ . This means that the geodesics on  $\mathbb{R}P^m$  are exactly the images of geodesics on  $S^m$ . This shows that the real projective spaces are complete.

**Definition 7.17.** Let  $(M, g)$  be a Riemannian manifold and  $\gamma : I \rightarrow M$  be a  $C^r$ -curve on  $M$ . A **variation** of  $\gamma$  is a  $C^r$ -map

$$\Phi : (-\epsilon, \epsilon) \times I \rightarrow M$$

such that for all  $s \in I$ ,  $\Phi_0(s) = \Phi(0, s) = \gamma(s)$ . If the interval is compact i.e. of the form  $I = [a, b]$ , then the variation  $\Phi$  is called **proper** if for all  $t \in (-\epsilon, \epsilon)$ ,  $\Phi_t(a) = \gamma(a)$  and  $\Phi_t(b) = \gamma(b)$ .

**Definition 7.18.** Let  $(M, g)$  be a Riemannian manifold and  $\gamma : I \rightarrow M$  be a  $C^2$ -curve on  $M$ . For every compact interval  $[a, b] \subset I$  we define the energy functional  $E_{[a,b]}$  by

$$E_{[a,b]}(\gamma) = \frac{1}{2} \int_a^b g(\dot{\gamma}(t), \dot{\gamma}(t)) dt.$$

A  $C^2$ -curve  $\gamma : I \rightarrow M$  is called a **critical point** for the energy functional if every proper variation  $\Phi$  of  $\gamma|_{[a,b]}$  satisfies

$$\frac{d}{dt}(E_{[a,b]}(\Phi_t))|_{t=0} = 0.$$

We shall now prove that geodesics can be characterized as the critical points of the energy functional.

**Theorem 7.19.** *A  $C^2$ -curve  $\gamma : I = [a, b] \rightarrow M$  is a critical point for the energy functional if and only if it is a geodesic.*

PROOF. For a  $C^2$ -map  $\Phi : (-\epsilon, \epsilon) \times I \rightarrow M$ ,  $\Phi : (t, s) \mapsto \Phi(t, s)$  we define the vector fields  $X = d\Phi(\partial/\partial s)$  and  $Y = d\Phi(\partial/\partial t)$  along  $\Phi$ . The following shows that the vector fields  $X$  and  $Y$  commute.

$$\begin{aligned} \nabla_X Y - \nabla_Y X &= [X, Y] \\ &= [d\Phi(\partial/\partial s), d\Phi(\partial/\partial t)] \\ &= d\Phi([\partial/\partial s, \partial/\partial t]) \\ &= 0, \end{aligned}$$

since  $[\partial/\partial s, \partial/\partial t] = 0$ . We now assume that  $\Phi$  is a proper variation of  $\gamma$ . Then

$$\begin{aligned} \frac{d}{dt}(E_{[a,b]}(\Phi_t)) &= \frac{1}{2} \frac{d}{dt} \left( \int_a^b g(X, X) ds \right) \\ &= \frac{1}{2} \int_a^b \frac{d}{dt} (g(X, X)) ds \\ &= \int_a^b g(\nabla_Y X, X) ds \\ &= \int_a^b g(\nabla_X Y, X) ds \\ &= \int_a^b \left( \frac{d}{ds} (g(Y, X)) - g(Y, \nabla_X X) \right) ds \\ &= [g(Y, X)]_a^b - \int_a^b g(Y, \nabla_X X) ds. \end{aligned}$$

The variation is proper, so  $Y(t, a) = Y(t, b) = 0$ . Furthermore  $X(0, s) = \partial\Phi/\partial s(0, s) = \dot{\gamma}(s)$ , so

$$\frac{d}{dt}(E_{[a,b]}(\Phi_t))|_{t=0} = - \int_a^b g(Y(0, s), (\nabla_{\dot{\gamma}}\dot{\gamma})(s)) ds.$$

The last integral vanishes for every proper variation  $\Phi$  of  $\gamma$  if and only if  $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$ .  $\square$

A geodesic  $\gamma : I \rightarrow (M, g)$  is a special case of what is called a **harmonic map**  $\phi : (M, g) \rightarrow (N, h)$  between Riemannian manifolds. Other examples are conformal immersions  $\psi : (M^2, g) \rightarrow (N, h)$  which parametrize the so called minimal surfaces in  $(N, h)$ . For a reference on harmonic maps see H. Urakawa, *Calculus of Variations and Harmonic Maps*, Translations of Mathematical Monographs **132**, AMS (1993).

Let  $(M^m, g)$  be an  $m$ -dimensional Riemannian manifold,  $p \in M$  and

$$S_p^{m-1} = \{v \in T_p M \mid g_p(v, v) = 1\}$$

be the unit sphere in the tangent space  $T_p M$  at  $p$ . Then every point  $w \in T_p M \setminus \{0\}$  can be written as  $w = r_w \cdot v_w$ , where  $r_w = |w|$  and  $v_w = w/|w| \in S_p^{m-1}$ . For  $v \in S_p^{m-1}$  let  $\gamma_v : (-\alpha_v, \beta_v) \rightarrow M$  be the maximal geodesic such that  $\alpha_v, \beta_v \in \mathbb{R}^+ \cup \{\infty\}$ ,  $\gamma_v(0) = p$  and  $\dot{\gamma}_v(0) = v$ . It can be shown that the real number

$$\epsilon_p = \inf\{\alpha_v, \beta_v \mid v \in S_p^{m-1}\}$$

is positive so the open ball

$$B_{\epsilon_p}^m(0) = \{v \in T_p M \mid g_p(v, v) < \epsilon_p^2\}$$

is non-empty. The **exponential map**  $\exp_p : B_{\epsilon_p}^m(0) \rightarrow M$  at  $p$  is defined by

$$\exp_p : w \mapsto \begin{cases} p & \text{if } w = 0 \\ \gamma_{v_w}(r_w) & \text{if } w \neq 0. \end{cases}$$

Note that for  $v \in S_p^{m-1}$  the line segment  $\lambda_v : (-\epsilon_p, \epsilon_p) \rightarrow T_p M$  with  $\lambda_v : t \mapsto t \cdot v$  is mapped onto the geodesic  $\gamma_v$  i.e. locally we have  $\gamma_v = \exp_p \circ \lambda_v$ . One can prove that the map  $\exp_p$  is smooth and it follows from its definition that the differential

$$d(\exp_p)_0 : T_p M \rightarrow T_p M$$

is the identity map for the tangent space  $T_p M$ . Then the inverse mapping theorem tells us that there exists an  $r_p \in \mathbb{R}^+$  such that if  $U_p = B_{r_p}^m(0)$  and  $V_p = \exp_p(U_p)$  then  $\exp_p|_{U_p} : U_p \rightarrow V_p$  is a diffeomorphism parametrizing the open subset  $V_p$  of  $M$ .

The next result shows that the geodesics are locally the shortest paths between their endpoints.

**Theorem 7.20.** *Let  $(M, g)$  be a Riemannian manifold. Then the geodesics are locally the shortest paths between their end points.*

PROOF. Let  $p \in M$ ,  $U = B_r^m(0)$  in  $T_pM$  and  $V = \exp_p(U)$  be such that the restriction

$$\phi = \exp_p|_U : U \rightarrow V$$

of the exponential map at  $p$  is a diffeomorphism. We define a metric  $\tilde{g}$  on  $U$  such that for each  $X, Y \in C^\infty(TU)$  we have

$$\tilde{g}(X, Y) = g(d\phi(X), d\phi(Y)).$$

This turns  $\phi : (U, \tilde{g}) \rightarrow (V, g)$  into an isometry. It then follows from the construction of the exponential map, that the geodesics in  $(U, \tilde{g})$  through the point  $0 = \phi^{-1}(p)$  are exactly the lines  $\lambda_v : t \mapsto t \cdot v$  where  $v \in T_pM$ .

Now let  $q \in B_r^m(0) \setminus \{0\}$  and  $\lambda_q : [0, 1] \rightarrow B_r^m(0)$  be the curve  $\lambda_q : t \mapsto t \cdot q$ . Further let  $\sigma : [0, 1] \rightarrow U$  be any  $C^1$ -curve such that  $\sigma(0) = 0$  and  $\sigma(1) = q$ . Along the curve  $\sigma$  we define the vector field  $X$  with  $X : t \mapsto \sigma(t)$  and the tangent field  $\dot{\sigma} : t \mapsto \dot{\sigma}(t)$  to  $\sigma$ . Then the radial component  $\dot{\sigma}_{\text{rad}}$  of  $\dot{\sigma}$  is the orthogonal projection of  $\dot{\sigma}$  onto the line generated by  $X$  i.e.

$$\dot{\sigma}_{\text{rad}} : t \mapsto \frac{\tilde{g}(\dot{\sigma}(t), X(t))}{\tilde{g}(X(t), X(t))} X(t).$$

Then it is easily checked that

$$|\dot{\sigma}_{\text{rad}}(t)| = \frac{|\tilde{g}(\dot{\sigma}(t), X(t))|}{|X(t)|}$$

and

$$\frac{d}{dt}|X(t)| = \frac{d}{dt}\sqrt{\tilde{g}(X(t), X(t))} = \frac{\tilde{g}(\dot{\sigma}(t), X(t))}{|X(t)|}.$$

Combining these two relations we yield

$$|\dot{\sigma}_{\text{rad}}(t)| \geq \frac{d}{dt}|X(t)|.$$

This means that

$$\begin{aligned} L(\sigma) &= \int_0^1 |\dot{\sigma}(t)| dt \\ &\geq \int_0^1 |\dot{\sigma}_{\text{rad}}(t)| dt \end{aligned}$$

$$\begin{aligned}
&\geq \int_0^1 \frac{d}{dt} |X(t)| dt \\
&= |X(1)| - |X(0)| \\
&= |q| \\
&= L(\lambda_q).
\end{aligned}$$

This proves that in fact  $\gamma$  is the shortest path connecting  $p$  and  $q$ .  $\square$

**Definition 7.21.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold with the induced metric  $g$ . Then the **mean curvature vector field** of  $M$  in  $N$  is the smooth section  $H : M \rightarrow NM$  of the normal bundle  $NM$  given by

$$H = \frac{1}{m} \text{trace } B = \frac{1}{m} \sum_{k=1}^m B(X_k, X_k).$$

Here  $B$  is the second fundamental form of  $M$  in  $N$  and  $\{X_1, \dots, X_m\}$  is any local orthonormal frame for the tangent bundle  $TM$  of  $M$ . The submanifold  $M$  is said to be **minimal** in  $N$  if  $H \equiv 0$  and **totally geodesic** in  $N$  if  $B \equiv 0$ .

**Proposition 7.22.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold equipped with the induced metric  $g$ . Then the following conditions are equivalent

- (i)  $M$  is totally geodesic in  $N$
- (ii) if  $\gamma : I \rightarrow M$  is a curve, then the following conditions are equivalent
  - (a)  $\gamma : I \rightarrow M$  is a geodesic in  $M$ ,
  - (b)  $\gamma : I \rightarrow M$  is a geodesic in  $N$ .

**PROOF.** The result is a direct consequence of the following decomposition formula

$$\nabla_{\dot{\gamma}} \dot{\gamma} = (\nabla_{\dot{\gamma}} \dot{\gamma})^{\top} + (\nabla_{\dot{\gamma}} \dot{\gamma})^{\perp} = \tilde{\nabla}_{\dot{\gamma}} \dot{\gamma} + B(\dot{\gamma}, \dot{\gamma}).$$

$\square$

**Proposition 7.23.** Let  $(N, h)$  be a Riemannian manifold and  $M$  be a complete submanifold of  $N$ . For a point  $(p, v)$  of the tangent bundle  $TM$  let  $\gamma_{(p,v)} : I \rightarrow N$  be the maximal geodesic in  $N$  with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . Then  $M$  is totally geodesic in  $(N, h)$  if and only if  $\gamma_{(p,v)}(I) \subset M$  for all  $(p, v) \in TM$ .

**PROOF.** See Exercise 7.3.  $\square$

**Corollary 7.24.** *Let  $(N, h)$  be a Riemannian manifold,  $p \in N$  and  $V$  be an  $m$ -dimensional linear subspace of the tangent space  $T_p N$  of  $N$  at  $p$ . Then there exists (locally) at most one totally geodesic submanifold  $M$  of  $(N, h)$  such that  $T_p M = V$ .*

PROOF. See Exercise 7.4. □

**Proposition 7.25.** *Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold of  $N$  which is the fixpoint set of an isometry  $\phi : N \rightarrow N$ . Then  $M$  is totally geodesic in  $N$ .*

PROOF. Let  $p \in M$ ,  $v \in T_p M$  and  $c : J \rightarrow M$  be a curve such that  $c(0) = p$  and  $\dot{c}(0) = v$ . Since  $M$  is the fix point set of  $\phi$  we have  $\phi(p) = p$  and  $d\phi_p(v) = v$ . Further let  $\gamma : I \rightarrow N$  be the maximal geodesic in  $N$  with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . The map  $\phi : N \rightarrow N$  is an isometry so the curve  $\phi \circ \gamma : I \rightarrow N$  is also a geodesic. The uniqueness result of Theorem 7.9,  $\phi(\gamma(0)) = \gamma(0)$  and  $d\phi(\dot{\gamma}(0)) = \dot{\gamma}(0)$  then imply that  $\phi(\gamma) = \gamma$ . Hence the image of the geodesic  $\gamma : I \rightarrow N$  is contained in  $M$ , so following Proposition 7.23 the submanifold  $M$  is totally geodesic in  $N$ . □

**Corollary 7.26.** *Let  $m < n$  be positive integers. Then the  $m$ -dimensional sphere*

$$S^m = \{(x, 0) \in \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} \mid |x|^2 = 1\}$$

*is a totally geodesic submanifold of*

$$S^n = \{(x, y) \in \mathbb{R}^{m+1} \times \mathbb{R}^{n-m} \mid |x|^2 + |y|^2 = 1\}.$$

PROOF. The statement is a direct consequence of the fact that  $S^m$  is the fixpoint set of the isometry  $\phi : S^n \rightarrow S^n$  of  $S^n$  with  $(x, y) \mapsto (x, -y)$ . □

**Corollary 7.27.** *Let  $m < n$  be positive integers. Let  $H^n$  be the  $n$ -dimensional hyperbolic space modelled on the upper half space  $\mathbb{R}^+ \times \mathbb{R}^{n-1}$  equipped with the Riemannian metric*

$$g(X, Y) = \frac{1}{x_1^2} \langle X, Y \rangle_{\mathbb{R}^n},$$

*where  $x = (x_1, \dots, x_n) \in H^n$ . Then the  $m$ -dimensional hyperbolic space*

$$H^m = \{(x, 0) \in H^n \mid x \in \mathbb{R}^m\}$$

*is totally geodesic in  $H^n$ .*

PROOF. See Exercise 7.6. □



**Definition 7.28.** A **symmetric space** is a Riemannian manifold  $(M, g)$  such that for each point  $p \in M$  there exists a global isometry  $\phi_p : M \rightarrow M$  which is a geodesic symmetry fixing  $p$ . By this we mean that  $\phi_p(p) = p$  and the tangent map  $d\phi_p : T_pM \rightarrow T_pM$  satisfies  $d\phi_p(X) = -X$  for all  $X \in T_pM$ .

**Example 7.29.** Let  $p$  be an arbitrary point on the standard sphere  $S^m$  as a subset of  $\mathbb{R}^{n+1}$ . Then the reflection  $\rho_p : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$  about the line generated by  $p$  is given by

$$\rho_p : q \mapsto 2\langle q, p \rangle p - q.$$

This is a linear map hence identical to its differential  $\rho_p : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^{n+1}$ . The restriction  $\phi_p = \rho_p|_{S^m} : S^m \rightarrow S^m$ , to the sphere, is an isometry that fixes  $p$ . Its tangent map  $d\phi_p : T_pS^m \rightarrow T_pS^m$  satisfies  $d\phi_p(X) = -X$  for all  $X \in T_pS^m$ . We have shown that the homogeneous space  $S^m$  is symmetric.

**Proposition 7.30.** *Every symmetric space is complete.*

PROOF. See Exercise 7.9. □

**Theorem 7.31.** *Let  $(M, g)$  be a complete Riemannian manifold. If  $p, q \in M$  then there exists a geodesic  $\gamma : \mathbb{R} \rightarrow M$  such that  $\gamma(0) = p$  and  $\gamma(1) = q$ .*

PROOF. See Exercise 7.10. □

**Theorem 7.32.** *Every symmetric space is homogeneous.*

PROOF. See Exercise 7.11. □

## Exercises

**Exercise 7.1.** The result of Exercise 5.3 shows that the two dimensional hyperbolic disc  $H^2$  introduced in Example 5.7 is isometric to the upper half plane  $M = (\{(x, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}^+\})$  equipped with the Riemannian metric

$$g(X, Y) = \frac{1}{y^2} \langle X, Y \rangle_{\mathbb{R}^2}.$$

Use your local library to find all geodesics in  $(M, g)$ .

**Exercise 7.2.** Let the orthogonal group  $\mathbf{O}(n)$  be equipped with the standard left-invariant metric

$$g(A, B) = \text{trace}(A^t B).$$

Prove that a  $C^2$ -curve  $\gamma : (-\epsilon, \epsilon) \rightarrow \mathbf{O}(n)$  is a geodesic if and only if

$$\gamma^t \cdot \ddot{\gamma} = \ddot{\gamma}^t \cdot \gamma.$$

**Exercise 7.3.** Find a proof of Proposition 7.23.

**Exercise 7.4.** Find a proof of Corollary 7.24.

**Exercise 7.5.** For the real parameter  $\theta \in (0, \pi/2)$  define the 2-dimensional torus  $T_\theta^2$  by

$$T_\theta^2 = \{(\cos \theta e^{i\alpha}, \sin \theta e^{i\beta}) \in S^3 \mid \alpha, \beta \in \mathbb{R}\}.$$

Determine for which  $\theta \in (0, \pi/2)$  the torus  $T_\theta^2$  is a minimal submanifold of the 3-dimensional sphere

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

**Exercise 7.6.** Find a proof of Corollary 7.27.

**Exercise 7.7.** Determine the totally geodesic submanifolds of the  $m$ -dimensional real projective space  $\mathbb{R}P^m$ .

**Exercise 7.8.** Let the orthogonal group  $\mathbf{O}(n)$  be equipped with the left-invariant metric

$$g(A, B) = \text{trace}(A^t B)$$

and let  $K$  be a Lie subgroup of  $\mathbf{O}(n)$ . Prove that  $K$  is totally geodesic in  $\mathbf{O}(n)$ .

**Exercise 7.9.** Find a proof of Proposition 7.30.

**Exercise 7.10.** Use your local library to find a proof of Theorem 7.31.

**Exercise 7.11.** Find a proof of Theorem 7.32.

## The Riemann Curvature Tensor

In this chapter we introduce the Riemann curvature tensor and the notion of sectional curvature of a Riemannian manifold. These generalize the Gaussian curvature playing a central role in classical differential geometry.

We prove that the Euclidean spaces, the standard spheres and the hyperbolic spaces all have constant sectional curvature. We determine the Riemannian curvature tensor for manifolds of constant sectional curvature and also for an important class of Lie groups. We then derive the important Gauss equation comparing the sectional curvatures of a submanifold and that of its ambient space.

Let  $(M, g)$  be a Riemannian manifold and  $\nabla$  be its Levi-Civita connection. Then to each vector field  $X \in C^\infty(TM)$  we have the **first order covariant derivative**

$$\nabla_X : C^\infty(TM) \rightarrow C^\infty(TM)$$

in the direction of  $X$  satisfying

$$\nabla_X : Z \mapsto \nabla_X Z.$$

We shall now generalize this and introduce the covariant derivative of tensor fields of type  $(r, 0)$  or  $(r, 1)$ .

As motivation, let us assume that  $A$  is a tensor field of type  $(2, 1)$ . If we differentiate  $A(Y, Z)$  in the direction of  $X$  applying the naive "product rule"

$$\nabla_X(A(Y, Z)) = (\nabla_X A)(Y, Z) + A(\nabla_X Y, Z) + A(Y, \nabla_X Z)$$

we get

$$(\nabla_X A)(Y, Z) = \nabla_X(A(Y, Z)) - A(\nabla_X Y, Z) - A(Y, \nabla_X Z),$$

where  $\nabla_X A$  is the "covariant derivative" of the tensor field  $A$  in the direction of  $X$ . This naive idea turns out to be very useful and leads to the following formal definition.

**Definition 8.1.** Let  $(M, g)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$ . For a tensor field  $A : C_r^\infty(TM) \rightarrow C_0^\infty(TM)$  of

type  $(r, 0)$  we define its **covariant derivative**

$$\nabla A : C_{r+1}^\infty(TM) \rightarrow C_0^\infty(TM)$$

by

$$\begin{aligned} \nabla A : (X, X_1, \dots, X_r) &\mapsto (\nabla_X A)(X_1, \dots, X_r) = \\ &X(A(X_1, \dots, X_r)) - \sum_{k=1}^r A(X_1, \dots, X_{k-1}, \nabla_X X_k, X_{k+1}, \dots, X_r). \end{aligned}$$

A tensor field  $A$  of type  $(r, 0)$  is said to be **parallel** if  $\nabla A \equiv 0$ .

The following result can be seen as, yet another, compatibility of the Levi-Civita connection  $\nabla$  of  $(M, g)$  with the Riemannian metric  $g$ .

**Proposition 8.2.** *Let  $(M, g)$  be a Riemannian manifold. Then the metric  $g$  is a parallel tensor field of type  $(2, 0)$ .*

PROOF. See Exercise 8.1. □

Let  $(M, g)$  be a Riemannian manifold. Then its Levi-Civita connection  $\nabla$  is tensorial in its first argument i.e. if  $X, Y \in C^\infty(TM)$  and  $f, g \in C^\infty(M)$  then

$$\nabla(fX + gY)Z = f\nabla_X Z + g\nabla_Y Z.$$

This means that a vector field  $Z \in C^\infty(TM)$  on  $M$  induces a **natural tensor field**  $\mathcal{Z} : C_1^\infty(TM) \rightarrow C_1^\infty(TM)$  of type  $(1, 1)$  given by

$$\mathcal{Z} : X \mapsto \nabla_X Z.$$

**Definition 8.3.** Let  $(M, g)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$ . For a tensor field  $B : C_r^\infty(TM) \rightarrow C_1^\infty(TM)$  of type  $(r, 1)$  we define its **covariant derivative**

$$\nabla B : C_{r+1}^\infty(TM) \rightarrow C_1^\infty(TM)$$

by

$$\begin{aligned} \nabla B : (X, X_1, \dots, X_r) &\mapsto (\nabla_X B)(X_1, \dots, X_r) = \\ &\nabla_X(B(X_1, \dots, X_r)) - \sum_{k=1}^r B(X_1, \dots, X_{k-1}, \nabla_X X_k, X_{k+1}, \dots, X_r). \end{aligned}$$

A tensor field  $B$  of type  $(r, 1)$  is said to be **parallel** if  $\nabla B \equiv 0$ .

**Definition 8.4.** Let  $X, Y \in C^\infty(TM)$  be two vector fields on the Riemannian manifold  $(M, g)$  with Levi-Civita connection  $\nabla$ . Then the **second order covariant derivative**

$$\nabla_{X, Y}^2 : C^\infty(TM) \rightarrow C^\infty(TM)$$

is defined by

$$\nabla^2_{X,Y}Z = (\nabla_X \mathcal{Z})(Y),$$

where  $\mathcal{Z}$  is the natural  $(1,1)$ -tensor field induced by  $Z \in C^\infty(TM)$ .

As a direct consequence of Definitions 8.3 and 8.4 we see that if  $X, Y, Z \in C^\infty(TM)$  then the second order covariant derivative  $\nabla^2_{X,Y}$  satisfies

$$\nabla^2_{X,Y}Z = \nabla_X(\mathcal{Z}(Y)) - \mathcal{Z}(\nabla_X Y) = \nabla_X \nabla_Y Z - \nabla_{\nabla_X Y} Z.$$

**Definition 8.5.** Let  $(M, g)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$ . Then we define its Riemann curvature operator

$$R : C^\infty(TM) \times C^\infty(TM) \times C^\infty(TM) \rightarrow C^\infty(TM)$$

as twice the skew-symmetric part of the second covariant derivative  $\nabla^2$  i.e.

$$R(X, Y)Z = \nabla^2_{X,Y}Z - \nabla^2_{Y,X}Z.$$

The next remarkable result shows that the curvature operator is a tensor field.

**Theorem 8.6.** Let  $(M, g)$  be a Riemannian manifold with Levi-Civita connection  $\nabla$ . Then the **curvature**  $R : C_3^\infty(TM) \rightarrow C_1^\infty(TM)$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]}Z$$

is a **tensor field** on  $M$  of type  $(3, 1)$ .

PROOF. See Exercise 8.2.  $\square$

The reader should note that the Riemann curvature tensor  $R$  is an intrinsic object since it only depends on the intrinsic Levi-Civita connection  $\nabla$ . The following result shows that the curvature tensor has many beautiful symmetries.

**Proposition 8.7.** Let  $(M, g)$  be a Riemannian manifold. For vector fields  $X, Y, Z, W \in C^\infty(TM)$  on  $M$  we then have

- (i)  $R(X, Y)Z = -R(Y, X)Z$ ,
- (ii)  $g(R(X, Y)Z, W) = -g(R(X, Y)W, Z)$ ,
- (iii)  $R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0$ ,
- (iv)  $g(R(X, Y)Z, W) = g(R(Z, W)X, Y)$ ,
- (v)  $6 \cdot R(X, Y)Z = R(X, Y + Z)(Y + Z) - R(X, Y - Z)(Y - Z) + R(X + Z, Y)(X + Z) - R(X - Z, Y)(X - Z)$ .

PROOF. See Exercise 8.3.  $\square$

Part (iii) of Proposition 8.7 is the so called first **Bianchi identity**. The second Bianchi identity is a similar result concerning the covariant derivative  $\nabla R$  of the curvature tensor. This will not be treated here.

Let  $(M, g)$  be a Riemannian manifold and  $p \in M$ . Then a **section**  $V$  at  $p$  is a 2-dimensional subspace of the tangent space  $T_p M$ . The set

$$G_2(T_p M) = \{V \mid V \text{ is a section of } T_p M\}$$

of sections is called the **Grassmannian** of 2-planes at  $p$ .

**Lemma 8.8.** *Let  $(M, g)$  be a Riemannian manifold,  $p \in M$  and  $X, Y, Z, W \in T_p M$  be tangent vectors at  $p$  such that the two sections  $\text{span}_{\mathbb{R}}\{X, Y\}$  and  $\text{span}_{\mathbb{R}}\{Z, W\}$  are identical. Then*

$$\frac{g(R(X, Y)Y, X)}{|X|^2|Y|^2 - g(X, Y)^2} = \frac{g(R(Z, W)W, Z)}{|Z|^2|W|^2 - g(Z, W)^2}.$$

PROOF. See Exercise 8.4. □

This leads to the following natural definition of the sectional curvature.

**Definition 8.9.** Let  $(M, g)$  be a Riemannian manifold and  $p \in M$ . Then the function  $K_p : G_2(T_p M) \rightarrow \mathbb{R}$  given by

$$K_p : \text{span}_{\mathbb{R}}\{X, Y\} \mapsto \frac{g(R(X, Y)Y, X)}{|X|^2|Y|^2 - g(X, Y)^2}$$

is called the **sectional curvature** at  $p$ . We often write  $K(X, Y)$  for  $K(\text{span}_{\mathbb{R}}\{X, Y\})$ .

**Definition 8.10.** Let  $(M, g)$  be a Riemannian manifold,  $p \in M$  and  $K_p : G_2(T_p M) \rightarrow \mathbb{R}$  be the sectional curvature at  $p$ . Then we define the functions  $\delta, \Delta : M \rightarrow \mathbb{R}$  by

$$\delta : p \mapsto \min_{V \in G_2(T_p M)} K_p(V) \quad \text{and} \quad \Delta : p \mapsto \max_{V \in G_2(T_p M)} K_p(V).$$

The Riemannian manifold  $(M, g)$  is said to be

- (i) of **positive curvature** if  $\delta(p) \geq 0$  for all  $p$ ,
- (ii) of **strictly positive curvature** if  $\delta(p) > 0$  for all  $p$ ,
- (iii) of **negative curvature** if  $\Delta(p) \leq 0$  for all  $p$ ,
- (iv) of **strictly negative curvature** if  $\Delta(p) < 0$  for all  $p$ ,
- (v) of **constant curvature** if  $\delta = \Delta$  is constant,
- (vi) **flat** if  $\delta \equiv \Delta \equiv 0$ .

The next result shows how the curvature tensor can be expressed in terms of local coordinates.

**Proposition 8.11.** *Let  $(M, g)$  be a Riemannian manifold and let  $(U, x)$  be local coordinates on  $M$ . For  $i, j, k, l = 1, \dots, m$  put*

$$X_i = \frac{\partial}{\partial x_i}, \quad g_{ij} = g(X_i, X_j) \quad \text{and} \quad R_{ijkl} = g(R(X_i, X_j)X_k, X_l).$$

Then

$$R_{ijkl} = \sum_{s=1}^m g_{sl} \left( \frac{\partial \Gamma_{jk}^s}{\partial x_i} - \frac{\partial \Gamma_{ik}^s}{\partial x_j} + \sum_{r=1}^m \{ \Gamma_{jk}^r \cdot \Gamma_{ir}^s - \Gamma_{ik}^r \cdot \Gamma_{jr}^s \} \right),$$

where  $\Gamma_{ij}^k$  are the Christoffel symbols of the Levi-Civita connection  $\nabla$  of  $(M, g)$  with respect to  $(U, x)$ .

PROOF. Using the fact that  $[X_i, X_j] = 0$  we obtain

$$\begin{aligned} R(X_i, X_j)X_k &= \nabla_{X_i} \nabla_{X_j} X_k - \nabla_{X_j} \nabla_{X_i} X_k \\ &= \sum_{s=1}^m \{ \nabla_{X_i} (\Gamma_{jk}^s \cdot X_s) - \nabla_{X_j} (\Gamma_{ik}^s \cdot X_s) \} \\ &= \sum_{s=1}^m \left( \frac{\partial \Gamma_{jk}^s}{\partial x_i} \cdot X_s + \sum_{r=1}^m \Gamma_{jk}^s \Gamma_{is}^r X_r - \frac{\partial \Gamma_{ik}^s}{\partial x_j} \cdot X_s - \sum_{r=1}^m \Gamma_{ik}^s \Gamma_{js}^r X_r \right) \\ &= \sum_{s=1}^m \left( \frac{\partial \Gamma_{jk}^s}{\partial x_i} - \frac{\partial \Gamma_{ik}^s}{\partial x_j} + \sum_{r=1}^m \{ \Gamma_{jk}^r \Gamma_{ir}^s - \Gamma_{ik}^r \Gamma_{jr}^s \} \right) X_s. \end{aligned}$$

□

For the  $m$ -dimensional vector space  $\mathbb{R}^m$  equipped with the Euclidean metric  $\langle \cdot, \cdot \rangle_{\mathbb{R}^m}$  the set  $\{\partial/\partial x_1, \dots, \partial/\partial x_m\}$  is a global frame for the tangent bundle  $T\mathbb{R}^m$ . In this situation we have  $g_{ij} = \delta_{ij}$ , so  $\Gamma_{ij}^k \equiv 0$  by Example 6.12. This implies that  $R \equiv 0$  so  $E^m$  is flat.

**Example 8.12.** The standard sphere  $S^m$  has constant sectional curvature  $+1$  (see Exercises 8.7 and 8.8) and the hyperbolic space  $H^m$  has constant sectional curvature  $-1$  (see Exercise 8.9).

Our next aim is a formula for the curvature tensor for manifolds of constant sectional curvature. This we present in Corollary 8.16. First we need some preparations.

**Lemma 8.13.** *Let  $(M, g)$  be a Riemannian manifold,  $p \in M$  and  $Y \in T_p M$ . Then the map  $\tilde{Y} : T_p M \rightarrow T_p M$  given by*

$$\tilde{Y} : X \mapsto R(X, Y)Y$$

*is a symmetric endomorphism of the tangent space  $T_p M$ .*

PROOF. For  $Z \in T_p M$  we have

$$\begin{aligned} g(\tilde{Y}(X), Z) &= g(R(X, Y)Y, Z) = g(R(Y, Z)X, Y) \\ &= g(R(Z, Y)Y, X) = g(X, \tilde{Y}(Z)). \end{aligned}$$

□

For a tangent vector  $Y \in T_p M$  with  $|Y| = 1$  let  $\mathcal{N}(Y)$  be the normal space to  $Y$

$$\mathcal{N}(Y) = \{X \in T_p M \mid g(X, Y) = 0\}.$$

The fact that  $\tilde{Y}(Y) = 0$  and Lemma 8.13 ensure the existence of an orthonormal basis of eigenvectors  $X_1, \dots, X_{m-1}$  for the restriction of the symmetric endomorphism  $\tilde{Y}$  to  $\mathcal{N}(Y)$ . Without loss of generality, we can assume that the corresponding eigenvalues satisfy

$$\lambda_1(p) \leq \dots \leq \lambda_{m-1}(p).$$

If  $X \in \mathcal{N}(Y)$ ,  $|X| = 1$  and  $\tilde{Y}(X) = \lambda X$  then

$$K_p(X, Y) = g(R(X, Y)Y, X) = g(\tilde{Y}(X), X) = \lambda.$$

This means that the eigenvalues satisfy

$$\delta(p) \leq \lambda_1(p) \leq \dots \leq \lambda_{m-1}(p) \leq \Delta(p).$$

**Definition 8.14.** Let  $(M, g)$  be a Riemannian manifold. Then define the smooth tensor field  $R_1 : C_3^\infty(TM) \rightarrow C_1^\infty(TM)$  of type  $(3, 1)$  by

$$R_1(X, Y)Z = g(Y, Z)X - g(X, Z)Y.$$

**Proposition 8.15.** Let  $(M, g)$  be a smooth Riemannian manifold and  $X, Y, Z$  be vector fields on  $M$ . Then

- (i)  $|R(X, Y)Y - \frac{\delta+\Delta}{2}R_1(X, Y)Y| \leq \frac{1}{2}(\Delta - \delta)|X||Y|^2$
- (ii)  $|R(X, Y)Z - \frac{\delta+\Delta}{2}R_1(X, Y)Z| \leq \frac{2}{3}(\Delta - \delta)|X||Y||Z|$

PROOF. Without loss of generality we can assume that  $|X| = |Y| = |Z| = 1$ . If  $X = X^\perp + X^\top$  with  $X^\perp \perp Y$  and  $X^\top$  is a multiple of  $Y$  then  $R(X, Y)Z = R(X^\perp, Y)Z$  and  $|X^\perp| \leq |X|$  so we can also assume that  $X \perp Y$ . Then  $R_1(X, Y)Y = \langle Y, Y \rangle X - \langle X, Y \rangle Y = X$ .

The first statement follows from the fact that the symmetric endomorphism of  $T_p M$  with

$$X \mapsto \left\{ R(X, Y)Y - \frac{\Delta + \delta}{2} \cdot X \right\}$$

restricted to  $\mathcal{N}(Y)$  has eigenvalues in the interval  $[\frac{\delta-\Delta}{2}, \frac{\Delta-\delta}{2}]$ .



It is easily checked that the operator  $R_1$  satisfies the conditions of Proposition 8.7 and hence  $D = R - \frac{\Delta + \delta}{2} \cdot R_1$  as well. This implies that

$$\begin{aligned} 6 \cdot D(X, Y)Z &= D(X, Y + Z)(Y + Z) - D(X, Y - Z)(Y - Z) \\ &+ D(X + Z, Y)(X + Z) - D(X - Z, Y)(X - Z). \end{aligned}$$

The second statement then follows from

$$\begin{aligned} 6|D(X, Y)Z| &\leq \frac{1}{2}(\Delta - \delta)\{|X|(|Y + Z|^2 + |Y - Z|^2) \\ &\quad + |Y|(|X + Z|^2 + |X - Z|^2)\} \\ &= \frac{1}{2}(\Delta - \delta)\{2|X|(|Y|^2 + |Z|^2) + 2|Y|(|X|^2 + |Z|^2)\} \\ &= 4(\Delta - \delta). \end{aligned}$$

□

As a direct consequence of Proposition 8.15 we have the following useful result.

**Corollary 8.16.** *Let  $(M, g)$  be a Riemannian manifold of constant curvature  $\kappa$ . Then the curvature tensor  $R$  is given by*

$$R(X, Y)Z = \kappa \cdot (g(Y, Z)X - g(X, Z)Y).$$

PROOF. The result follows directly from  $\kappa = \delta = \Delta$ . □

**Proposition 8.17.** *Let  $(G, g)$  be a Lie group equipped with a left-invariant metric such that for all  $X \in \mathfrak{g}$  the endomorphism*

$$\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$$

*is skew-symmetric with respect to  $g$ . Then for any left-invariant vector fields  $X, Y, Z \in \mathfrak{g}$  the curvature tensor  $R$  is given by*

$$R(X, Y)Z = -\frac{1}{4} [[X, Y], Z].$$

PROOF. See Exercise 8.6. □

We shall now prove the important **Gauss equation** comparing the curvature tensors of a submanifold and its ambient space in terms of the second fundamental form.

**Theorem 8.18.** *Let  $(N, h)$  be a Riemannian manifold and  $M$  be a submanifold of  $N$  equipped with the induced metric  $g$ . Let  $X, Y, Z, W \in C^\infty(TN)$  be vector fields extending  $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in C^\infty(TM)$ . Then*

$$\begin{aligned} &g(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W}) - h(R(X, Y)Z, W) \\ &= h(B(\tilde{Y}, \tilde{Z}), B(\tilde{X}, \tilde{W})) - h(B(\tilde{X}, \tilde{Z}), B(\tilde{Y}, \tilde{W})). \end{aligned}$$

PROOF. Using the definitions of the curvature tensors  $R$ ,  $\tilde{R}$ , the Levi-Civita connection  $\tilde{\nabla}$  and the second fundamental form of  $\tilde{M}$  in  $M$  we obtain

$$\begin{aligned}
& g(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W}) \\
&= g(\tilde{\nabla}_{\tilde{X}}\tilde{\nabla}_{\tilde{Y}}\tilde{Z} - \tilde{\nabla}_{\tilde{Y}}\tilde{\nabla}_{\tilde{X}}\tilde{Z} - \tilde{\nabla}_{[\tilde{X}, \tilde{Y}]}^{\tilde{Z}}, \tilde{W}) \\
&= h((\nabla_X(\nabla_Y Z - B(Y, Z)))^\top - (\nabla_Y(\nabla_X Z - B(X, Z)))^\top, W) \\
&\quad - h((\nabla_{[X, Y]} Z - B([X, Y], Z))^\top, W) \\
&= h(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z, W) \\
&\quad - h(\nabla_X(B(Y, Z)), W) + \nabla_Y(B(X, Z)), W) \\
&= h(R(X, Y)Z, W) \\
&\quad + h((B(Y, Z)), \nabla_X W) - h(B(X, Z)), \nabla_Y W) \\
&= h(R(X, Y)Z, W) \\
&\quad + h(B(Y, Z), B(X, W)) - h(B(X, Z), B(Y, W)).
\end{aligned}$$

□

We shall now apply the Gauss equation to the classical situation of a regular surface  $\Sigma$  as a submanifold of the 3-dimensional Euclidean space  $\mathbb{R}^3$ . Let  $\{\tilde{X}, \tilde{Y}\}$  be a local orthonormal frame for the tangent bundle  $T\Sigma$  of  $\Sigma$  around a point  $p \in \Sigma$  and  $\tilde{N}$  be the local Gauss map with  $\tilde{N} = \tilde{X} \times \tilde{Y}$ . If  $X, Y, N$  are local extensions of  $\tilde{X}, \tilde{Y}, \tilde{N}$  then the second fundamental form  $B$  of  $\Sigma$  in  $\mathbb{R}^3$  satisfies

$$\begin{aligned}
B(\tilde{X}, \tilde{Y}) &= \langle \partial_X Y, N \rangle N \\
&= -\langle Y, \partial_X N \rangle N \\
&= -\langle Y, dN(X) \rangle N \\
&= \langle Y, S_p(X) \rangle N,
\end{aligned}$$

where  $S_p : T_p \Sigma \rightarrow T_p \Sigma$  is the shape operator at  $p$ . If we now apply the fact that  $\mathbb{R}^3$  is flat, the Gauss equation tells us that the sectional curvature  $K(\tilde{X}, \tilde{Y})$  of  $\Sigma$  at  $p$  satisfies

$$\begin{aligned}
K(\tilde{X}, \tilde{Y}) &= \langle \tilde{R}(\tilde{X}, \tilde{Y})\tilde{Y}, \tilde{X} \rangle \\
&= \langle B(\tilde{Y}, \tilde{Y}), B(\tilde{X}, \tilde{X}) \rangle - \langle B(\tilde{X}, \tilde{Y}), B(\tilde{Y}, \tilde{X}) \rangle \\
&= \det S_p.
\end{aligned}$$

In other word, the sectional curvature  $K(\tilde{X}, \tilde{Y})$  is the determinant of the shape operator  $S_p$  at  $p$  i.e. **the classical Gaussian curvature**.

As a direct consequence of the Gauss equation we have the following useful result, see for example Exercises 8.8 and 8.9.

**Corollary 8.19.** *Let  $(N, h)$  be a Riemannian manifold and  $(M, g)$  be a totally geodesic submanifold of  $N$ . Let  $X, Y, Z, W \in C^\infty(TN)$  be vector fields extending  $\tilde{X}, \tilde{Y}, \tilde{Z}, \tilde{W} \in C^\infty(TM)$ . Then*

$$g(\tilde{R}(\tilde{X}, \tilde{Y})\tilde{Z}, \tilde{W}) = h(R(X, Y)Z, W).$$

We conclude this chapter by defining the Ricci and scalar curvatures of a Riemannian manifold. These are obtained by taking traces over the curvature tensor and play an important role in Riemannian geometry.

**Definition 8.20.** Let  $(M, g)$  be a Riemannian manifold, then

(i) the **Ricci operator**  $r : C_1^\infty(TM) \rightarrow C_1^\infty(M)$  is defined by

$$r(X) = \sum_{i=1}^m R(X, e_i)e_i,$$

(ii) the **Ricci curvature**  $Ric : C_2^\infty(TM) \rightarrow C_0^\infty(TM)$  by

$$Ric(X, Y) = \sum_{i=1}^m g(R(X, e_i)e_i, Y), \quad \text{and}$$

(iii) the **scalar curvature**  $s \in C^\infty(M)$  by

$$s = \sum_{j=1}^m Ric(e_j, e_j) = \sum_{j=1}^m \sum_{i=1}^m g(R(e_i, e_j)e_j, e_i).$$

Here  $\{e_1, \dots, e_m\}$  is any local orthonormal frame for the tangent bundle.

**Corollary 8.21.** *Let  $(M, g)$  be a Riemannian manifold of constant sectional curvature  $\kappa$ . Then the following holds*

$$s = m \cdot (m - 1) \cdot \kappa.$$

**PROOF.** Let  $\{e_1, \dots, e_m\}$  be an orthonormal basis, then Corollary 8.16 implies that

$$\begin{aligned} Ric(e_j, e_j) &= \sum_{i=1}^m g(R(e_j, e_i)e_i, e_j) \\ &= \sum_{i=1}^m g(\kappa(g(e_i, e_i)e_j - g(e_j, e_i)e_i), e_j) \\ &= \kappa \left( \sum_{i=1}^m g(e_i, e_i)g(e_j, e_j) - \sum_{i=1}^m g(e_i, e_j)g(e_i, e_j) \right) \end{aligned}$$

$$= \kappa \left( \sum_{i=1}^m 1 - \sum_{i=1}^m \delta_{ij} \right) = (m-1) \cdot \kappa.$$

To obtain the formula for the scalar curvature  $s$  we only have to multiply the constant Ricci curvature  $Ric(e_j, e_j)$  by  $m$ .  $\square$

For further reading on different notions of curvature we recommend the interesting book, Wolfgang Kühnel, Differential Geometry: Curves - Surfaces - Manifolds, AMS (2002).

## Exercises

**Exercise 8.1.** Let  $(M, g)$  be a Riemannian manifold. Prove that the tensor field  $g$  of type  $(2, 0)$  is parallel with respect to the Levi-Civita connection.

**Exercise 8.2.** Let  $(M, g)$  be a Riemannian manifold. Prove that the curvature  $R$  is a tensor field of type  $(3, 1)$ .

**Exercise 8.3.** Find a proof for Proposition 8.7.

**Exercise 8.4.** Find a proof for Lemma 8.8.

**Exercise 8.5.** Let  $\mathbb{R}^m$  and  $\mathbb{C}^m$  be equipped with their standard Euclidean metric  $g$  given by

$$g(z, w) = \operatorname{Re} \sum_{k=1}^m z_k \bar{w}_k$$

and let

$$T^m = \{z \in \mathbb{C}^m \mid |z_1| = \dots = |z_m| = 1\}$$

be the  $m$ -dimensional torus in  $\mathbb{C}^m$  with the induced metric. Find an isometric immersion  $\phi : \mathbb{R}^m \rightarrow T^m$ , determine all geodesics on  $T^m$  and prove that the torus is flat.

**Exercise 8.6.** Find a proof for Proposition 8.17.

**Exercise 8.7.** Let the Lie group  $S^3 \cong \mathbf{SU}(2)$  be equipped with the metric

$$g(Z, W) = \frac{1}{2} \operatorname{Re}(\operatorname{trace}(\bar{Z}^t W)).$$

- (i) Find an orthonormal basis for  $T_e \mathbf{SU}(2)$ .
- (ii) Prove that  $(\mathbf{SU}(2), g)$  has constant sectional curvature  $+1$ .

**Exercise 8.8.** Let  $S^m$  be the unit sphere in  $\mathbb{R}^{m+1}$  equipped with the standard Euclidean metric  $\langle \cdot, \cdot \rangle_{\mathbb{R}^{m+1}}$ . Use the results of Corollaries 7.26, 8.19 and Exercise 8.7 to prove that  $(S^m, \langle \cdot, \cdot \rangle_{\mathbb{R}^{m+1}})$  has constant sectional curvature  $+1$ .

**Exercise 8.9.** Let  $H^m$  be the  $m$ -dimensional hyperbolic space modelled on the upper half space  $\mathbb{R}^+ \times \mathbb{R}^{m-1}$  equipped with the Riemannian metric

$$g(X, Y) = \frac{1}{x_1^2} \langle X, Y \rangle_{\mathbb{R}^m},$$

where  $x = (x_1, \dots, x_m) \in H^m$ . For  $k = 1, \dots, m$  let the vector fields  $X_k \in C^\infty(TH^m)$  be given by

$$(X_k)_x = x_1 \cdot \frac{\partial}{\partial x_k}$$

and define the operation  $*$  on  $H^m$  by

$$(\alpha, x) * (\beta, y) = (\alpha \cdot \beta, \alpha \cdot y + x).$$

Prove that

- (i)  $(H^m, *)$  is a Lie group,
- (ii) the vector fields  $X_1, \dots, X_m$  are left-invariant,
- (iii)  $[X_k, X_l] = 0$  and  $[X_1, X_k] = X_k$  for  $k, l = 2, \dots, m$ ,
- (iv) the metric  $g$  is left-invariant,
- (v)  $(H^m, g)$  has constant curvature  $-1$ .

Compare with Exercises 6.4 and 7.1.

## Curvature and Local Geometry

This chapter is devoted to the study of the local geometry of Riemannian manifolds and how this is controlled by the curvature tensor. For this we introduce the notion of a Jacobi field which is a useful tool in differential geometry. With this at hand we yield a fundamental comparison result describing the curvature dependence of local distances.

Let  $(M, g)$  be a Riemannian manifold. By a smooth **1-parameter family of geodesics** we mean a  $C^\infty$ -map

$$\Phi : (-\epsilon, \epsilon) \times I \rightarrow M$$

such that the curve  $\gamma_t : I \rightarrow M$  given by  $\gamma_t : s \mapsto \Phi(t, s)$  is a geodesic for all  $t \in (-\epsilon, \epsilon)$ . The variable  $t \in (-\epsilon, \epsilon)$  is called the **family parameter** of  $\Phi$ .

**Proposition 9.1.** *Let  $(M, g)$  be a Riemannian manifold and  $\Phi : (-\epsilon, \epsilon) \times I \rightarrow M$  be a 1-parameter family of geodesics. Then for each  $t \in (-\epsilon, \epsilon)$  the vector field  $J_t : I \rightarrow C^\infty(TM)$  along  $\gamma_t$  given by*

$$J_t(s) = \frac{\partial \Phi}{\partial t}(t, s)$$

*satisfies the second order ordinary differential equation*

$$\nabla_{\dot{\gamma}_t} \nabla_{\dot{\gamma}_t} J_t + R(J_t, \dot{\gamma}_t) \dot{\gamma}_t = 0.$$

**PROOF.** Along  $\Phi$  we put  $X(t, s) = \partial \Phi / \partial s$  and  $J(t, s) = \partial \Phi / \partial t$ . The fact that  $[\partial / \partial t, \partial / \partial s] = 0$  implies that

$$[J, X] = [d\Phi(\partial / \partial t), d\Phi(\partial / \partial s)] = d\Phi([\partial / \partial t, \partial / \partial s]) = 0.$$

Since  $\Phi$  is a family of geodesics we have  $\nabla_X X = 0$  and the definition of the curvature tensor then gives

$$\begin{aligned} R(J, X)X &= \nabla_J \nabla_X X - \nabla_X \nabla_J X - \nabla_{[J, X]} X \\ &= -\nabla_X \nabla_J X \\ &= -\nabla_X \nabla_X J. \end{aligned}$$

Hence for each  $t \in (-\epsilon, \epsilon)$  we have

$$\nabla_{\dot{\gamma}_t} \nabla_{\dot{\gamma}_t} J_t + R(J_t, \dot{\gamma}_t) \dot{\gamma}_t = 0.$$

□

**Definition 9.2.** Let  $(M, g)$  be a Riemannian manifold,  $\gamma : I \rightarrow M$  be a geodesic and  $X = \dot{\gamma}$ . A  $C^2$  vector field  $J$  along  $\gamma$  is called a **Jacobi field** if

$$(5) \quad \nabla_X \nabla_X J + R(J, X)X = 0$$

along  $\gamma$ . We denote the space of all Jacobi fields along  $\gamma$  by  $\mathcal{J}_\gamma(TM)$ .

We shall now give an example of a 1-parameter family of geodesics in the  $(m+1)$ -dimensional Euclidean space  $E^{m+1}$ .

**Example 9.3.** Let  $c, n : \mathbb{R} \rightarrow E^{m+1}$  be smooth curves such that the image  $n(\mathbb{R})$  of  $n$  is contained in the unit sphere  $S^m$ . If we define a map  $\Phi : \mathbb{R} \times \mathbb{R} \rightarrow E^{m+1}$  by

$$\Phi : (t, s) \mapsto c(t) + s \cdot n(t)$$

then for each  $t \in \mathbb{R}$  the curve  $\gamma_t : s \mapsto \Phi(t, s)$  is a straight line and hence a geodesic in  $E^{m+1}$ . By differentiating with respect to the family parameter  $t$  we yield the Jacobi field  $J \in \mathcal{J}_{\gamma_0}(TE^{m+1})$  along  $\gamma_0$  with

$$J(s) = \frac{d}{dt} \Phi(t, s)|_{t=0} = \dot{c}(0) + s \cdot \dot{n}(0).$$

The Jacobi equation (5) on a Riemannian manifold is linear in  $J$ . This means that the space of Jacobi fields  $\mathcal{J}_\gamma(TM)$  along  $\gamma$  is a vector space. We are now interested in determining the dimension of this space

**Proposition 9.4.** Let  $\gamma : I \rightarrow M$  be a geodesic,  $0 \in I$ ,  $p = \gamma(0)$  and  $X = \dot{\gamma}$  along  $\gamma$ . If  $v, w \in T_p M$  are two tangent vectors at  $p$  then there exists a unique Jacobi field  $J$  along  $\gamma$ , such that  $J_p = v$  and  $(\nabla_X J)_p = w$ .

PROOF. Let  $\{X_1, \dots, X_m\}$  be an orthonormal frame of parallel vector fields along  $\gamma$ , see Proposition 7.8. If  $J$  is a vector field along  $\gamma$ , then

$$J = \sum_{i=1}^m a_i X_i$$

where  $a_i = g(J, X_i)$  are smooth functions on  $I$ . The vector fields  $X_1, \dots, X_m$  are parallel so

$$\nabla_X J = \sum_{i=1}^m \dot{a}_i X_i \quad \text{and} \quad \nabla_X \nabla_X J = \sum_{i=1}^m \ddot{a}_i X_i.$$



For the curvature tensor we have

$$R(X_i, X)X = \sum_{k=1}^m b_i^k X_k,$$

where  $b_i^k = g(R(X_i, X)X, X_k)$  are smooth functions on  $I$  depending on the geometry of  $(M, g)$ . This means that  $R(J, X)X$  is given by

$$R(J, X)X = \sum_{i,k=1}^m a_i b_i^k X_k.$$

and that  $J$  is a Jacobi field if and only if

$$\sum_{i=1}^m (\ddot{a}_i + \sum_{k=1}^m a_k b_k^i) X_i = 0.$$

This is equivalent to the second order system

$$\ddot{a}_i + \sum_{k=1}^m a_k b_k^i = 0 \quad \text{for all } i = 1, 2, \dots, m$$

of linear ordinary differential equations in  $a = (a_1, \dots, a_m)$ . A global solution will always exist and is uniquely determined by  $a(0)$  and  $\dot{a}(0)$ . This implies that  $J$  exists globally and is uniquely determined by the initial conditions

$$J(0) = v \quad \text{and} \quad (\nabla_X J)(0) = w.$$

□

**Corollary 9.5.** *Let  $(M, g)$  be an  $m$ -dimensional Riemannian manifold and  $\gamma : I \rightarrow M$  be a geodesic in  $M$ . Then the vector space  $\mathcal{J}_\gamma(TM)$  of all Jacobi fields along  $\gamma$  has the dimension  $2m$ .*

The following Lemma shows that when proving results about Jacobi fields along a geodesic  $\gamma$  we can always assume, without loss of generality, that  $|\dot{\gamma}| = 1$ .

**Lemma 9.6.** *Let  $(M, g)$  be a Riemannian manifold,  $\gamma : I \rightarrow M$  be a geodesic and  $J$  be a Jacobi field along  $\gamma$ . If  $\lambda \in \mathbb{R}^*$  and  $\sigma : \lambda I \rightarrow I$  is given by  $\sigma : t \mapsto t/\lambda$ , then  $\gamma \circ \sigma : \lambda I \rightarrow M$  is a geodesic and  $J \circ \sigma$  is a Jacobi field along  $\gamma \circ \sigma$ .*

PROOF. See Exercise 9.1. □

Next we determine the Jacobi fields which are tangential to a given geodesic.

**Proposition 9.7.** *Let  $(M, g)$  be a Riemannian manifold,  $\gamma : I \rightarrow M$  be a geodesic with  $|\dot{\gamma}| = 1$  and  $J$  be a Jacobi field along  $\gamma$ . Let  $J^\top$  be the tangential part of  $J$  given by*

$$J^\top = g(J, \dot{\gamma})\dot{\gamma} \quad \text{and} \quad J^\perp = J - J^\top$$

*be its normal part. Then  $J^\top$  and  $J^\perp$  are Jacobi fields along  $\gamma$  and there exist  $a, b \in \mathbb{R}$  such that  $J^\top(s) = (as + b)\dot{\gamma}(s)$  for all  $s \in I$ .*

PROOF. We now have

$$\begin{aligned} \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J^\top + R(J^\top, \dot{\gamma})\dot{\gamma} &= \nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}(g(J, \dot{\gamma})\dot{\gamma}) + R(g(J, \dot{\gamma})\dot{\gamma}, \dot{\gamma})\dot{\gamma} \\ &= g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J, \dot{\gamma})\dot{\gamma} \\ &= -g(R(J, \dot{\gamma})\dot{\gamma}, \dot{\gamma})\dot{\gamma} \\ &= 0. \end{aligned}$$

This shows that the tangential part  $J^\top$  of  $J$  is a Jacobi field. The fact that  $\mathcal{J}_\gamma(TM)$  is a vector space implies that the normal part  $J^\perp = J - J^\top$  of  $J$  also is a Jacobi field.

By differentiating  $g(J, \dot{\gamma})$  twice along  $\gamma$  we obtain

$$\frac{d^2}{ds^2}(g(J, \dot{\gamma})) = g(\nabla_{\dot{\gamma}}\nabla_{\dot{\gamma}}J, \dot{\gamma}) = -g(R(J, \dot{\gamma})\dot{\gamma}, \dot{\gamma}) = 0$$

so  $g(J, \dot{\gamma}(s)) = (as + b)$  for some  $a, b \in \mathbb{R}$ .  $\square$

**Corollary 9.8.** *Let  $(M, g)$  be a Riemannian manifold,  $\gamma : I \rightarrow M$  be a geodesic and  $J$  be a Jacobi field along  $\gamma$ . If*

$$g(J(t_0), \dot{\gamma}(t_0)) = 0 \quad \text{and} \quad g((\nabla_{\dot{\gamma}}J)(t_0), \dot{\gamma}(t_0)) = 0$$

*for some  $t_0 \in I$ , then  $g(J(t), \dot{\gamma}(t)) = 0$  for all  $t \in I$ .*

PROOF. This is a direct consequence of the fact that the function  $g(J, \dot{\gamma})$  satisfies the second order ordinary differential equation  $\ddot{f} = 0$  and the initial conditions  $f(0) = 0$  and  $\dot{f}(0) = 0$ .  $\square$

Our next aim is to show that if the Riemannian manifold  $(M, g)$  has constant sectional curvature then we can solve the Jacobi equation

$$\nabla_X\nabla_XJ + R(J, X)X = 0$$

along any given geodesic  $\gamma : I \rightarrow M$ . For this we introduce the following notation. For a real number  $\kappa \in \mathbb{R}$  we define the  $c_\kappa, s_\kappa : \mathbb{R} \rightarrow \mathbb{R}$  by

$$c_\kappa(s) = \begin{cases} \cosh(\sqrt{|\kappa|}s) & \text{if } \kappa < 0, \\ 1 & \text{if } \kappa = 0, \\ \cos(\sqrt{\kappa}s) & \text{if } \kappa > 0. \end{cases}$$

and

$$s_\kappa(s) = \begin{cases} \sinh(\sqrt{|\kappa|}s)/\sqrt{|\kappa|} & \text{if } \kappa < 0, \\ s & \text{if } \kappa = 0, \\ \sin(\sqrt{\kappa}s)/\sqrt{\kappa} & \text{if } \kappa > 0. \end{cases}$$

It is a well known fact that the unique solution to the initial value problem

$$\ddot{f} + \kappa \cdot f = 0, \quad f(0) = a \quad \text{and} \quad \dot{f}(0) = b$$

is the function  $f : \mathbb{R} \rightarrow \mathbb{R}$  satisfying  $f(s) = ac_\kappa(s) + bs_\kappa(s)$ .

**Example 9.9.** Let  $\mathbb{C}$  be the complex plane with the standard Euclidean metric  $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$  of constant sectional curvature  $\kappa = 0$ . The rotations about the origin produce a 1-parameter family of geodesics  $\Phi_t : s \mapsto se^{it}$ . Along the geodesic  $\gamma_0 : s \mapsto s$  we get the Jacobi field

$$J_0(s) = \frac{\partial \Phi_t}{\partial t}(0, s) = is$$

with  $|J_0(s)| = |s| = |s_\kappa(s)|$ .

**Example 9.10.** Let  $S^2$  be the unit sphere in the standard Euclidean 3-space  $\mathbb{C} \times \mathbb{R}$  with the induced metric of constant sectional curvature  $\kappa = +1$ . Rotations about the  $\mathbb{R}$ -axis produce a 1-parameter family of geodesics  $\Phi_t : s \mapsto (\sin(s)e^{it}, \cos(s))$ . Along the geodesic  $\gamma_0 : s \mapsto (\sin(s), \cos(s))$  we get the Jacobi field

$$J_0(s) = \frac{\partial \Phi_t}{\partial t}(0, s) = (i\sin(s), 0)$$

with  $|J_0(s)|^2 = \sin^2(s) = |s_\kappa(s)|^2$ .

**Example 9.11.** Let  $B_1^2(0)$  be the open unit disk in the complex plane with the hyperbolic metric

$$g(X, Y) = \frac{4}{(1 - |z|^2)^2} \langle \cdot, \cdot \rangle_{\mathbb{R}^2}$$

of constant sectional curvature  $\kappa = -1$ . Rotations about the origin produce a 1-parameter family of geodesics  $\Phi_t : s \mapsto \tanh(s/2)e^{it}$ . Along the geodesic  $\gamma_0 : s \mapsto \tanh(s/2)$  we get the Jacobi field

$$J_0(s) = \frac{\partial \Phi_t}{\partial t}(0, s) = i \cdot \tanh(s/2)$$

with

$$|J_0(s)|^2 = \frac{4 \cdot \tanh^2(s/2)}{(1 - \tanh^2(s/2))^2} = \sinh^2(s) = |s_\kappa(s)|^2.$$

**Example 9.12.** Let  $(M, g)$  be a Riemannian manifold of constant sectional curvature  $\kappa$  and  $\gamma : I \rightarrow M$  be a geodesic with  $|X| = 1$  where  $X = \dot{\gamma}$ . Further let  $P_1, P_2, \dots, P_{m-1}$  be parallel vector fields along  $\gamma$  such that  $g(P_i, P_j) = \delta_{ij}$  and  $g(P_i, X) = 0$ . Any vector field  $J$  along  $\gamma$  may now be written as

$$J(s) = \sum_{i=1}^{m-1} f_i(s)P_i(s) + f_m(s)X(s).$$

This means that  $J$  is a Jacobi field if and only if

$$\begin{aligned} \sum_{i=1}^{m-1} \ddot{f}_i(s)P_i(s) + \ddot{f}_m(s)X(s) &= \nabla_X \nabla_X J \\ &= -R(J, X)X \\ &= -R(J^\perp, X)X \\ &= -\kappa(g(X, X)J^\perp - g(J^\perp, X)X) \\ &= -\kappa J^\perp \\ &= -\kappa \sum_{i=1}^{m-1} f_i(s)P_i(s). \end{aligned}$$

This is equivalent to the following system of ordinary differential equations

$$(6) \quad \ddot{f}_m(s) = 0 \quad \text{and} \quad \ddot{f}_i(s) + \kappa f_i(s) = 0 \quad \text{for all } i = 1, 2, \dots, m-1.$$

It is clear that for the initial values

$$\begin{aligned} J(s_0) &= \sum_{i=1}^{m-1} v_i P_i(s_0) + v_m X(s_0), \\ (\nabla_X J)(s_0) &= \sum_{i=1}^{m-1} w_i P_i(s_0) + w_m X(s_0) \end{aligned}$$

or equivalently

$$f_i(s_0) = v_i \quad \text{and} \quad \dot{f}_i(s_0) = w_i \quad \text{for all } i = 1, 2, \dots, m$$

we have a unique and explicit solution to the system (6) on the whole of  $I$ . They are given by

$$f_m(s) = v_m + s w_m \quad \text{and} \quad f_i(s) = v_i c_\kappa(s) + w_i s_\kappa(s)$$

for all  $i = 1, 2, \dots, m-1$ . It should be noted that if  $g(J, X) = 0$  and  $J(0) = 0$  then

$$|J(s)| = |(\nabla_X J)(0)|_{s_\kappa(s)}.$$

In the next example we give a complete description of the Jacobi fields along a geodesic on the 2-dimensional sphere.

**Example 9.13.** Let  $S^2$  be the unit sphere in the standard Euclidean 3-space  $\mathbb{C} \times \mathbb{R}$  with the induced metric of constant curvature  $\kappa = +1$  and  $\gamma : \mathbb{R} \rightarrow S^2$  be the geodesic given by  $\gamma : s \mapsto (e^{is}, 0)$ . Then  $\dot{\gamma}(s) = (ie^{is}, 0)$  so it follows from Proposition (9.7) that all Jacobi fields tangential to  $\gamma$  are given by

$$J_{(a,b)}^T(s) = (as + b)(ie^{is}, 0) \quad \text{for some } a, b \in \mathbb{R}.$$

The vector field  $P : \mathbb{R} \rightarrow TS^2$  given by  $s \mapsto ((e^{is}, 0), (0, 1))$  satisfies  $\langle P, \dot{\gamma} \rangle = 0$  and  $|P| = 1$ . The sphere  $S^2$  is 2-dimensional and  $\dot{\gamma}$  is parallel along  $\gamma$  so  $P$  must be parallel. This implies that all the Jacobi fields orthogonal to  $\dot{\gamma}$  are given by

$$J_{(a,b)}^N(s) = (0, a \cos s + b \sin s) \quad \text{for some } a, b \in \mathbb{R}.$$

In more general situations, where we do not have constant curvature the exponential map can be used to produce Jacobi fields as follows. Let  $(M, g)$  be a complete Riemannian manifold,  $p \in M$  and  $v, w \in T_p M$ . Then  $s \mapsto s(v + tw)$  defines a 1-parameter family of lines in the tangent space  $T_p M$  which all pass through the origin  $0 \in T_p M$ . Remember that the exponential map

$$\exp_p|_{B_{\varepsilon_p}^m(0)} : B_{\varepsilon_p}^m(0) \rightarrow \exp_p(B_{\varepsilon_p}^m(0))$$

maps lines in  $T_p M$  through the origin onto geodesics on  $M$ . Hence the map

$$\Phi_t : s \mapsto \exp_p(s(v + tw))$$

is a 1-parameter family of geodesics through  $p \in M$ , as long as  $s(v + tw)$  is an element of  $B_{\varepsilon_p}^m(0)$ . This means that

$$J(s) = \frac{\partial \Phi_t}{\partial t}(t, s)|_{t=0} = d(\exp_p)_{s(v+tw)}(sw)|_{t=0} = d(\exp_p)_{sv}(sw)$$

is a Jacobi field along the geodesic  $\gamma : s \mapsto \Phi_0(s)$  with  $\gamma(0) = p$  and  $\dot{\gamma}(0) = v$ . Here

$$d(\exp_p)_{s(v+tw)} : T_{s(v+tw)}T_p M \rightarrow T_{\exp_p(s(v+tw))}M$$

is the linear tangent map of the exponential map  $\exp_p$  at  $s(v + tw)$ . Now differentiating with respect to the parameter  $s$  gives

$$(\nabla_X J)(0) = \frac{d}{ds}(d(\exp_p)_{sv}(sw))|_{s=0} = d(\exp_p)_0(w) = w.$$

The above calculations show that

$$J(0) = 0 \quad \text{and} \quad (\nabla_X J)(0) = w.$$

The following technical result is needed for the proof of the main Theorem 9.15 at the end of this chapter.

**Lemma 9.14.** *Let  $(M, g)$  be a Riemannian manifold with sectional curvature uniformly bounded above by  $\Delta$  and  $\gamma : [0, \alpha] \rightarrow M$  be a geodesic on  $M$  with  $|X| = 1$  where  $X = \dot{\gamma}$ . Further let  $J : [0, \alpha] \rightarrow TM$  be a Jacobi field along  $\gamma$  such that  $g(J, X) = 0$  and  $|J| \neq 0$  on  $(0, \alpha)$ . Then*

- (i)  $d^2(|J|)/ds^2 + \Delta \cdot |J| \geq 0$ ,
- (ii) if  $f : [0, \alpha] \rightarrow \mathbb{R}$  is a  $C^2$ -function such that
  - (a)  $\dot{f} + \Delta \cdot f = 0$  and  $f > 0$  on  $(0, \alpha)$ ,
  - (b)  $f(0) = |J(0)|$ , and
  - (c)  $\dot{f}(0) = |\nabla_X J(0)|$ ,
 then  $f(s) \leq |J(s)|$  on  $(0, \alpha)$ ,
- (iii) if  $J(0) = 0$ , then  $|\nabla_X J(0)| \cdot s_\Delta(s) \leq |J(s)|$  for all  $s \in (0, \alpha)$ .

PROOF. (i) Using the facts that  $|X| = 1$  and  $\langle X, J \rangle = 0$  we obtain

$$\begin{aligned}
 \frac{d^2}{ds^2}(|J|) &= \frac{d^2}{ds^2} \sqrt{g(J, J)} = \frac{d}{ds} \left( \frac{g(\nabla_X J, J)}{|J|} \right) \\
 &= \frac{g(\nabla_X \nabla_X J, J)}{|J|} + \frac{|\nabla_X J|^2 |J|^2 - g(\nabla_X J, J)^2}{|J|^3} \\
 &\geq \frac{g(\nabla_X \nabla_X J, J)}{|J|} \\
 &= -\frac{g(R(J, X)X, J)}{|J|} \\
 &= -K(X, J) \cdot |J| \\
 &\geq -\Delta \cdot |J|.
 \end{aligned}$$

(ii) Define the function  $h : [0, \alpha] \rightarrow \mathbb{R}$  by

$$h(s) = \begin{cases} \frac{|J(s)|}{f(s)} & \text{if } s \in (0, \alpha), \\ \lim_{s \rightarrow 0} \frac{|J(s)|}{f(s)} = 1 & \text{if } s = 0. \end{cases}$$

Then

$$\begin{aligned}
 \dot{h}(s) &= \frac{1}{f^2(s)} \left( \frac{d}{ds} (|J(s)|) f(s) - |J(s)| \dot{f}(s) \right) \\
 &= \frac{1}{f^2(s)} \int_0^s \frac{d}{dt} \left( \frac{d}{dt} (|J(t)|) f(t) - |J(t)| \dot{f}(t) \right) dt \\
 &= \frac{1}{f^2(s)} \int_0^s \left( \frac{d^2}{dt^2} (|J(t)|) f(t) - |J(t)| \ddot{f}(t) \right) dt
 \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{f^2(s)} \int_0^s f(t) \left( \frac{d^2}{dt^2} (|J(t)|) + \Delta \cdot |J(t)| \right) dt \\
&\geq 0.
\end{aligned}$$

This implies that  $\dot{h}(s) \geq 0$  so  $f(s) \leq |J(s)|$  for all  $s \in (0, \alpha)$ .

(iii) The function  $f(s) = |(\nabla_X J)(0)| \cdot s_\Delta(s)$  satisfies the differential equation

$$\ddot{f}(s) + \Delta f(s) = 0$$

and the initial conditions  $f(0) = |J(0)| = 0$ ,  $\dot{f}(0) = |(\nabla_X J)(0)|$  so it follows from (ii) that  $|(\nabla_X J)(0)| \cdot s_\Delta(s) = f(s) \leq |J(s)|$ .  $\square$

Let  $(M, g)$  be a Riemannian manifold of sectional curvature which is uniformly bounded above, i.e. there exists a  $\Delta \in \mathbb{R}$  such that  $K_p(V) \leq \Delta$  for all  $V \in G_2(T_p M)$  and  $p \in M$ . Let  $(M_\Delta, g_\Delta)$  be another Riemannian manifold which is complete and of constant sectional curvature  $K \equiv \Delta$ . Let  $p \in M$ ,  $p_\Delta \in M_\Delta$  and identify  $T_p M \cong \mathbb{R}^m \cong T_{p_\Delta} M_\Delta$ .

Let  $U$  be an open neighbourhood of  $\mathbb{R}^m$  around 0 such that the exponential maps  $(\exp)_p$  and  $(\exp)_{p_\Delta}$  are diffeomorphisms from  $U$  onto their images  $(\exp)_p(U)$  and  $(\exp)_{p_\Delta}(U)$ , respectively. Let  $(r, p, q)$  be a geodesic triangle i.e. a triangle with sides which are shortest paths between their endpoints. Furthermore let  $c : [a, b] \rightarrow M$  be the geodesic connecting  $r$  and  $q$  and  $v : [a, b] \rightarrow T_p M$  be the curve defined by  $c(t) = (\exp)_p(v(t))$ . Put  $c_\Delta(t) = (\exp)_{p_\Delta}(v(t))$  for  $t \in [a, b]$  and then it directly follows that  $c(a) = r$  and  $c(b) = q$ . Finally put  $r_\Delta = c_\Delta(a)$  and  $q_\Delta = c_\Delta(b)$ .

**Theorem 9.15.** *For the above situation the following inequality for the distance function  $d$  is satisfied*

$$d(q_\Delta, r_\Delta) \leq d(q, r).$$

PROOF. Define a 1-parameter family  $s \mapsto s \cdot v(t)$  of straight lines in  $T_p M$  through  $p$ . Then

$$\Phi_t : s \mapsto (\exp)_p(s \cdot v(t)) \quad \text{and} \quad \Phi_t^\Delta : s \mapsto (\exp)_{p_\Delta}(s \cdot v(t))$$

are 1-parameter families of geodesics through  $p \in M$ , and  $p_\Delta \in M_\Delta$ , respectively. Hence

$$J_t = \partial \Phi_t / \partial t \quad \text{and} \quad J_t^\Delta = \partial \Phi_t^\Delta / \partial t$$

are Jacobi fields satisfying the initial conditions

$$J_t(0) = J_t^\Delta(0) = 0 \quad \text{and} \quad (\nabla_X J_t)(0) = (\nabla_X J_t^\Delta)(0) = \dot{v}(t).$$

Using Lemma 9.14 we now obtain

$$|\dot{c}_\Delta(t)| = |J_t^\Delta(1)|$$

$$\begin{aligned}
&= |(\nabla_X J_t^\Delta)(0)| \cdot s_\Delta(1) \\
&= |(\nabla_X J_t)(0)| \cdot s_\Delta(1) \\
&\leq |J_t(1)| \\
&= |\dot{c}(t)|
\end{aligned}$$

The curve  $c$  is the shortest path between  $r$  and  $q$  so we have

$$d(r_\Delta, q_\Delta) \leq L(c_\Delta) \leq L(c) = d(r, q).$$

□

We now add the assumption that the sectional curvature of the manifold  $(M, g)$  is uniformly bounded below i.e. there exists a  $\delta \in \mathbb{R}$  such that  $\delta \leq K_p(V)$  for all  $V \in G_2(T_p M)$  and  $p \in M$ . Let  $(M_\delta, g_\delta)$  be a complete Riemannian manifold of constant sectional curvature  $\delta$ . Let  $p \in M$  and  $p_\delta \in M_\delta$  and identify  $T_p M \cong \mathbb{R}^m \cong T_{p_\delta} M_\delta$ . Then a similar construction as above gives two pairs of points  $q, r \in M$  and  $q_\delta, r_\delta \in M_\delta$  and shows that

$$d(q, r) \leq d(q_\delta, r_\delta).$$

Combining these two results we obtain **locally**

$$d(q_\Delta, r_\Delta) \leq d(q, r) \leq d(q_\delta, r_\delta).$$



## Exercises

**Exercise 9.1.** Find a proof of Lemma 9.6.

**Exercise 9.2.** Let  $(M, g)$  be a Riemannian manifold and  $\gamma : I \rightarrow M$  be a geodesic such that  $X = \dot{\gamma} \neq 0$ . Further let  $J$  be a non-vanishing Jacobi field along  $\gamma$  with  $g(X, J) = 0$ . Prove that if  $g(J, J)$  is constant along  $\gamma$  then  $(M, g)$  does not have strictly negative curvature.