mathcentre

Trigonometric functions

mc-TY-trig-2009-1

The sine, cosine and tangent of an angle are all defined in terms of trigonometry, but they can also be expressed as functions. In this unit we examine these functions and their graphs. We also see how to restrict the domain of each function in order to define an inverse function.

In order to master the techniques explained here it is vital that you undertake plenty of practice exercises so that they become second nature.

After reading this text, and/or viewing the video tutorial on this topic, you should be able to:

- specify the domain and the range of the three trigonometric functions $f(x) = \sin x$, $f(x) = \cos x$ and $f(x) = \tan x$,
- understand the difference between each function expressed in degrees and the corresponding function expressed in radians,
- express the periodicity of each function in either degrees or radians,
- specify a suitable restriction for the domain of each function so that an inverse function can be defined,
- find the appropriate value of x (in either degrees or radians) when given a value of $\sin x$, $\cos x$ or $\tan x$.

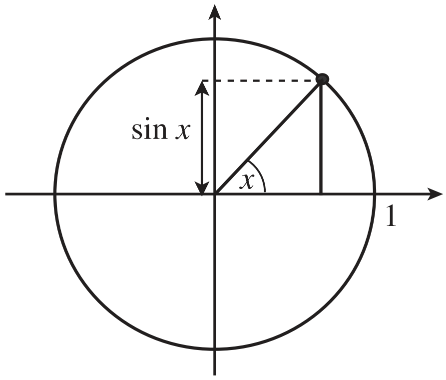
Contents

1. Introduction

In this unit we shall use information about the trigonometric ratios sine, cosine and tangent to define functions $f(x) = \sin x$, $f(x) = \cos x$ and $f(x) = \tan x$.

2. The sine function $f(x) = \sin x$

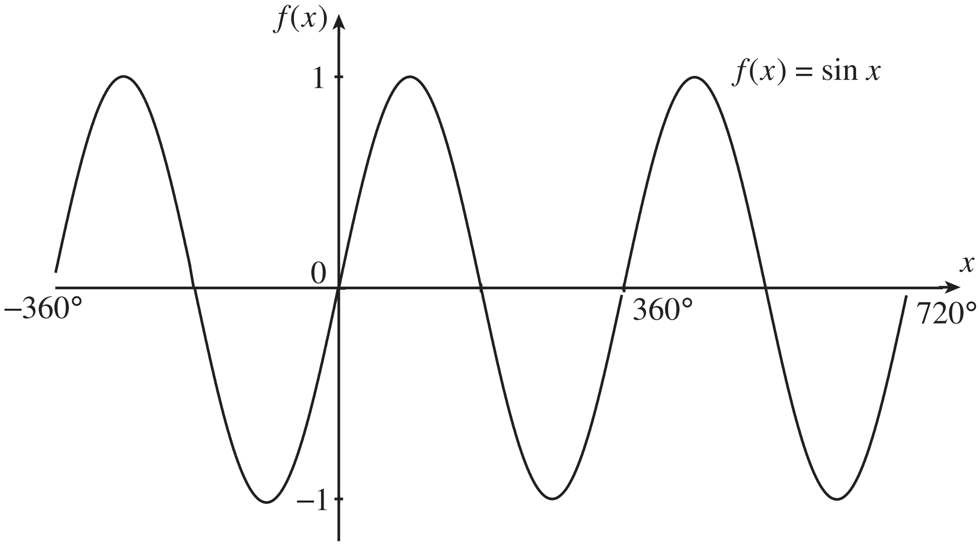
We shall start with the sine function, $f(x) = \sin x$. This function can be defined for any number x using a diagram like this.



We take a circle with centre at the origin, and with radius 1. We then draw a line from the origin, at x degrees from the horizontal axis, until it meets the circle, so that the line has length 1. We then look at the vertical axis coordinate of the point where the line and the circle meet, to find the value of $\sin x$.

The information from this picture can also be used to see how changing x affects the value of $\sin x$. We can use a table of values to plot selected points between $x = 0°$ and $x = 360°$, and draw a smooth curve between them. We can then extend the graph to the right and to the left, because we know that the graph repeats itself.

x	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$\sin x$	0	0.71	1	0.71	0	−0.71	−1	−0.71	0



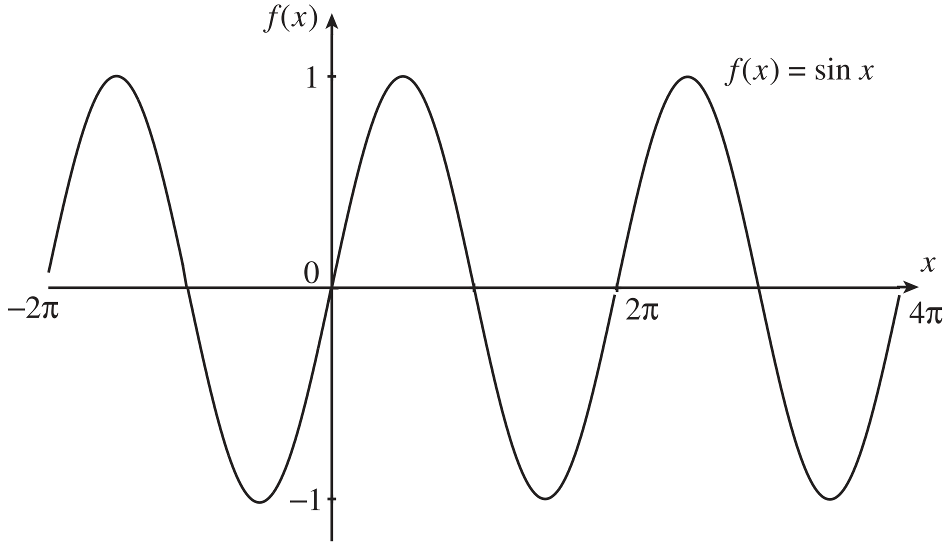
When $x = 0$, $\sin x = 0$. As we increase x to $90°$, $\sin x$ increases to 1. As we increase x further, $\sin x$ decreases. It becomes zero when $x = 180°$. It then continues to decrease, and becomes -1 when x is $270°$. After that $\sin x$ increases and becomes zero again when x reaches $360°$. We have now come back to where we started on the circle, so as we increase x further the cycle repeats.

We can also use this picture to see what happens when x is less than zero. If we decrease x from zero, $\sin x$ decreases. It becomes -1 when $x = -90°$. Then it becomes zero at $x = -180°$, and 1 at $x = -270°$. It then decreases and becomes zero when $x = -360°$. This cycle is repeated if we decrease x further.

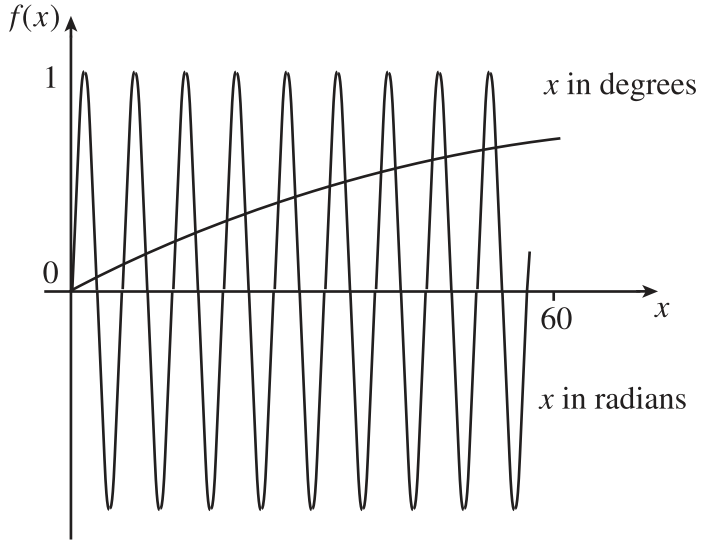
From this picture we can see that, whatever value we pick for x, the value of $\sin x$ must always be between -1 and 1. So the domain of $f(x) = \sin x$ contains all the real numbers, but the range is $-1 \leq \sin x \leq 1$. We can also see that the function repeats itself every $360°$. We can say that $\sin x = \sin(x + 360°)$. We say the function is periodic, with periodicity $360°$.

Sometimes we will want to work in radians instead of degrees. If we have $\sin x$ in radians, it is usually very different from $\sin x$ in degrees. For example $\sin 90° = 1$ but in radians $\sin(90)$ is about 0.894. We can use a table of values like the one we had before to plot a graph of $\sin x$ in radians. As 2π radians is the same as $360°$ the graph will be very similar to the graph for x in degrees, but now the labels on the axes have changed.

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
$\sin x$	0	0.71	1	0.71	0	-0.71	-1	-0.71	0



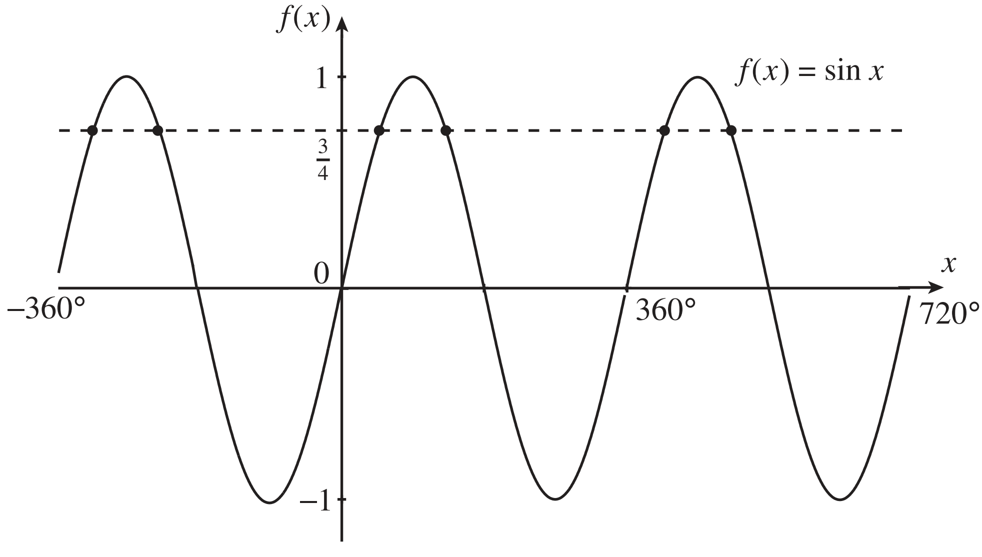
To compare the two graphs, we can keep the same scale on the x-axis and plot both graphs.



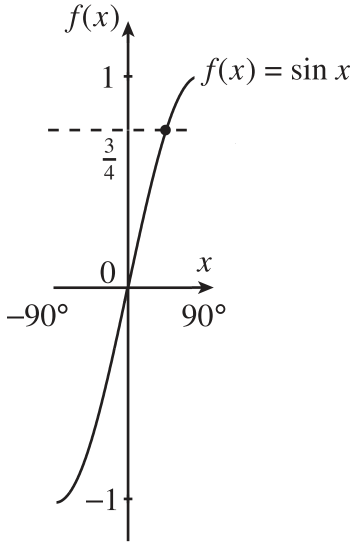
With x in degrees, the function $f(x) = \sin x$ has not reached 1 by the right-hand side of the graph, but with x in radians the function has oscillated several times. So these are quite different functions.

Sometimes, instead of finding the sine of an angle, we want to work backwards. We want to find an angle whose sine is, say, $\frac{3}{4}$. So we want to define a new function to give us the inverse sine of the number. We want to find a function such that $f^{-1}(x) = y$ whenever $f(y) = x$. In our case, we want $\sin x = \frac{3}{4}$, so that we shall want to have $\sin^{-1}(\frac{3}{4}) = x$.

Now this might seem to be a problem at first because, if we look back at our graph, we see that there are lots of angles with $\sin x = \frac{3}{4}$.



We cannot define a function to tell us what the inverse sine of $\frac{3}{4}$ should be if there is a choice of values for $f^{-1}(x)$. To get around this problem, we need to restrict the domain of our function $f(x) = \sin x$ so that we have only a part of the graph that gives us one angle for each sine value. This happens if we cut our domain down to $-90° \leq x \leq 90°$, or $-\pi \leq x \leq \pi$ if we work in radians.



We say that our function $f(x) = \sin x$ has domain $-90° \leq x \leq 90°$ and that it has an inverse, $f^{-1}(x) = \sin^{-1} x$. This inverse function is also written as $\arcsin x$. So, if the angle x lies in the range $-90° \leq x \leq 90°$ and $\sin x = \frac{3}{4}$, we say $x = \sin^{-1}(\frac{3}{4})$. You can use your calculator to work out inverse sines.

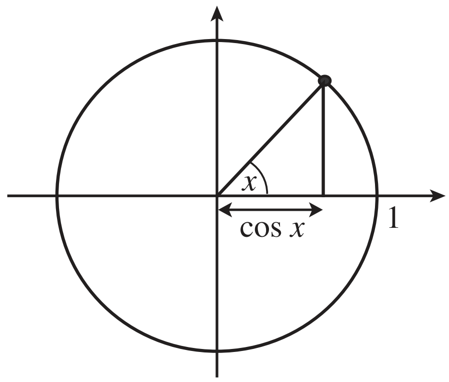
Key Point

The function $f(x) = \sin x$ has all real numbers in its domain, but its range is $-1 \leq \sin x \leq 1$. The values of the sine function are different, depending on whether the angle is in degrees or radians. The function is periodic with periodicity 360 degrees or 2π radians.

We can define an inverse function, denoted $f(x) = \sin^{-1} x$ or $f(x) = \arcsin x$, by restricting the domain of the sine function.

3. The cosine function $f(x) = \cos x$

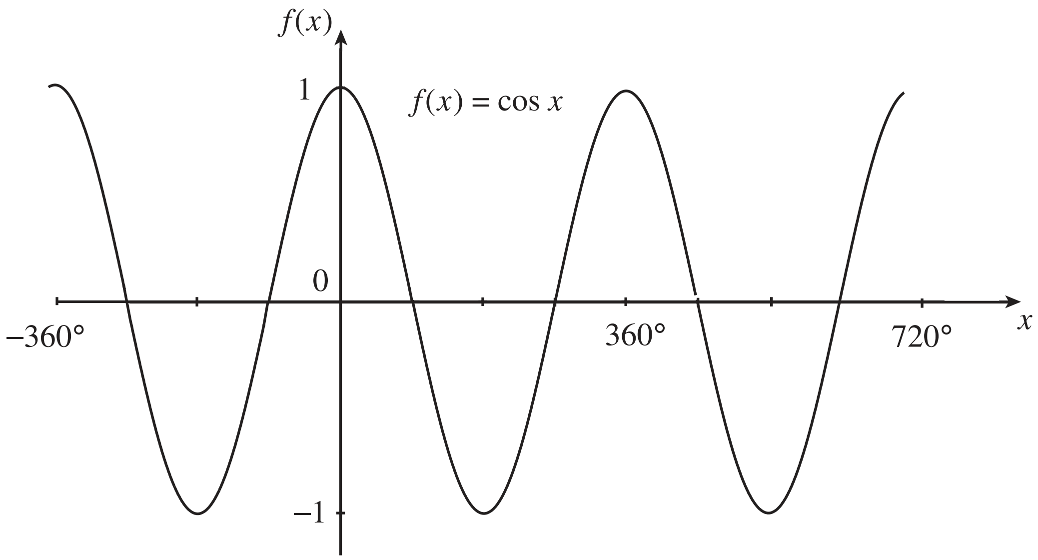
We shall now look at the cosine function, $f(x) = \cos x$. This function can be defined for any number x using a diagram like this.



We take a circle diagram similar to the one we used for the sine function. But now we look at the horizontal axis coordinate of the point where the line and the circle meet, to find the value of $\cos x$.

The information from this picture can also be used to see how changing x affects the value of $\cos x$. We can use a table of values to plot selected points between $x = 0°$ and $x = 360°$, and draw a smooth curve between them. We can then extend the graph to the right and to the left, because we know that the graph repeats itself.

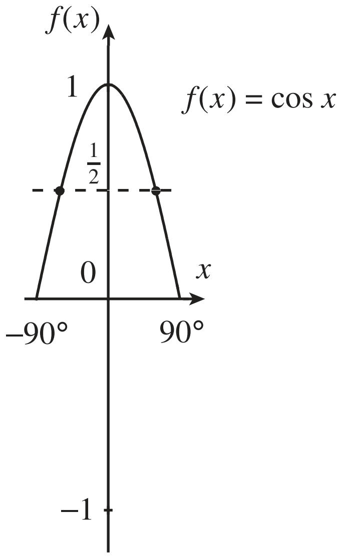
x	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$\cos x$	1	0.71	0	-0.71	-1	-0.71	0	0.71	1



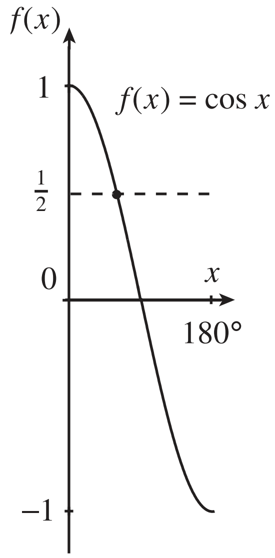
Again you can see that $\cos x$ must lie between -1 and 1. This function also has periodicity $360°$, or 2π if we work in radians.

As with $\sin x$, we should like to define an inverse function to tell us the angle having a cosine of, say, $\frac{1}{2}$. Unless we restrict the domain of $\cos x$, there will be many angles that could be $\cos^{-1}\frac{1}{2}$.

When we defined the inverse sine function, we restricted the domain of $\sin x$ to $-90° \leq x \leq 90°$. Let us see what happens if we do this for $\cos x$.



In this case we still have two angles for some values of the cosine function. Also, if we look at negative values then then there are no angles at all. So we need to choose a different domain. A good choice is $0° \leq x \leq 180°$, because this has only one angle giving each possible cosine value. So if we restrict the domain of $f(x) = \cos x$ in this way, we can define $\cos^{-1} x$.



So now we say that our function $f(x) = \cos x$ has domain $0° \leq x \leq 180°$ and that it has an inverse, $f^{-1}(x) = \cos^{-1} x$. This inverse function is also written as $\arccos x$. So, if the angle x lies in the range $0° \leq x \leq 180°$ and $\cos x = \frac{1}{2}$, we say $x = \cos^{-1}(\frac{1}{2})$.

Key Point

The function $f(x) = \cos x$ has all real numbers in its domain, but its range is $-1 \leq \cos x \leq 1$. The values of the cosine function are different, depending on whether the angle is in degrees or radians. The function is periodic with periodicity 360 degrees or 2π radians.

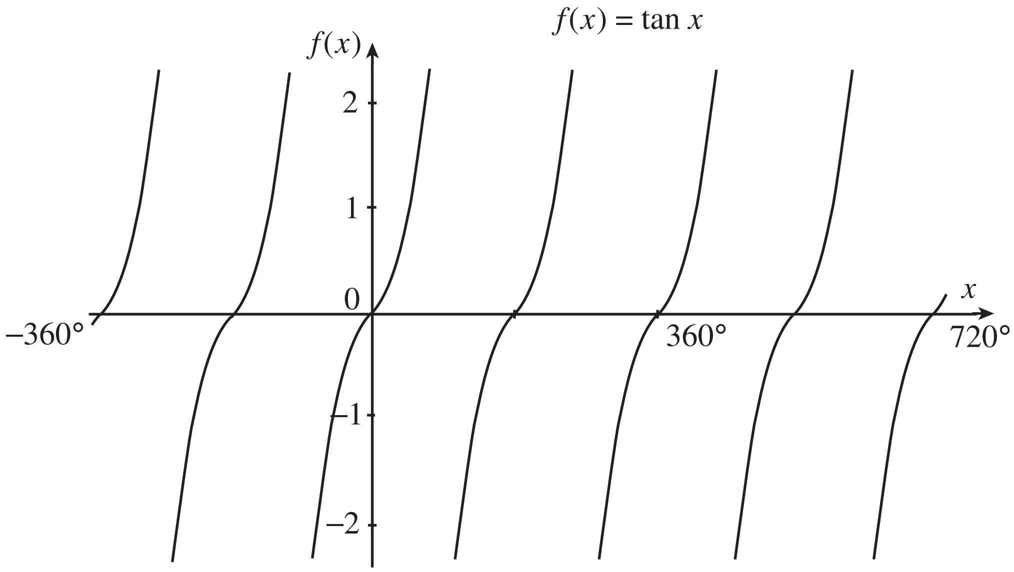
We can define an inverse function, denoted $f(x) = \cos^{-1} x$ or $f(x) = \arccos x$, by restricting the domain of the cosine function to $0° \leq x \leq 180°$ or $0 \leq x \leq \pi$.

4. The tangent function $f(x) = \tan x$

Finally we deal with $\tan x$, which is just $\sin x / \cos x$. We can use a table of values to plot selected points between $x = 0°$ and $x = 360°$, as before. But now we can use the values for $\sin x$ and $\cos x$ that we have already found.

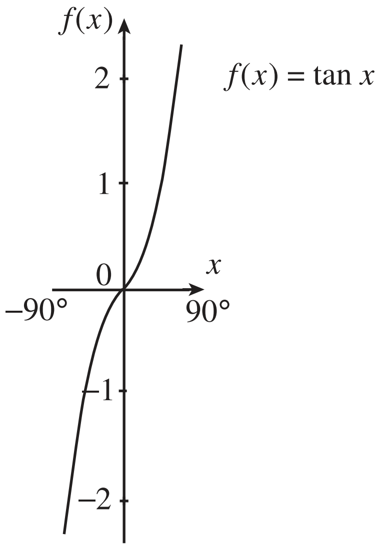
x	$0°$	$45°$	$90°$	$135°$	$180°$	$225°$	$270°$	$315°$	$360°$
$\sin x$	0	0.71	1	0.71	0	-0.71	-1	-0.71	0
$\cos x$	1	0.71	0	-0.71	-1	-0.71	0	0.71	1
$\tan x$	0	1	*	-1	0	1	*	-1	0

You can see from the table that there are some values of x for which $\cos x = 0$, and so $\tan x$ is not defined for these values of x. These are at $90°$, $270°$, and also other values differing from them' by multiples of $180°$.



Notice that, when we get to $x = 360°$, the graphs of $\sin x$ and $\cos x$ repeat themselves. As $\tan x$ depends on only $\sin x$ and $\cos x$, the graph of $\tan x$ must also repeat itself. But $\tan x$ repeats itself more often than $\sin x$ and $\cos x$. It repeats itself every $180°$. So $\tan x$ has a periodicity of $180°$, or π if you work in radians. Notice also that, unlike $\sin x$ and $\cos x$, the function $\tan x$ does not have to lie between -1 and 1. In fact $\tan x$ can take any value.

We can also define an inverse tangent function, and to do this we must restrict the domain of $f(x) = \tan x$. In fact we need to think twice as hard this time because, as well as making sure that we get only one angle giving each tangent value, we must also avoid trying to define $f(x) = \tan x$ over a region where there is a zero cosine value. So we cannot define $f(x) = \tan x$ for $0° \leq x \leq 180°$ because at $x = 90°$ there is a point where $f(x)$ is not defined. We get around this by instead restricting our domain to $-90° < x < 90°$, excluding the values $-90°$ and $90°$ themselves. This domain gives one angle for every tangent value, but does not include points where the tangent function is undefined.



Key Point

The function $f(x) = \tan x$ has all real numbers except odd multiples of 90° in its domain (in the case where x is expressed in degrees), or all real numbers except odd multiples of $\pi/2$ (in the case where x is expressed in radians. The range of the tangent function contains all real numbers. The function is periodic with periodicity 180 degrees or π radians.

We can define an inverse function, denoted $f(x) = \tan^{-1} x$ or $f(x) = \arctan x$, by restricting the domain of the tangent function to $-90^\circ < x < 90^\circ$ or $-\pi/2 < x < \pi/2$.

Exercises

(In these exercises, all angles are expressed in radians.)

1. State whether each of the following functions is periodic. If the function is periodic, give its periodicity.

(a) $\sin 3x$, (b) $1 + \tan x$, (c) $\cos(x + 1)$, (d) $\cos(x^2)$, (e) $\cos^2 x$,
(f) $\sin x + \cos x$, (g) $x + \sin x$.

2. Find the domain and range of the following functions:

(a) $\sin^{-1} x$, (b) $\cos^{-1} x$, (c) $\tan^{-1} x$.

Answers

1.

(a) $2\pi/3$ (b) π (c) 2π (d) not periodic
(e) π (f) 2π (g) not periodic

2.

(a) domain: $-1 \le x \le 1$, range: $-\pi/2 \le y \le \pi/2$;
(b) domain: $-1 \le x \le 1$, range: $0 \le y \le \pi$;
(c) domain: all real x, range: $-\pi/2 < y < \pi/2$.

