Differential Equations

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Abstract

These are notes for an undergraduate course on differential equations; please send corrections, suggestions and notes to courses@suchideas.com The author's homepage for all courses may be found on his website at SuchIdeas.com, which is where updated and corrected versions of these notes can also be found.

Thanks go to Professor G. Worster for allowing me to use his Differential Equations course (Michaelmas 2009) as the basis for these notes.

Contents

Prerequisites

A grasp of standard calculus (integration and differentiation).

1 Non-Rigorous Background

This course is about the study of differential equations, in which variables are investigated in terms of rates of change (not just with respect to time). It is, obviously, an area of mathematics with many direct applications to physics, including mechanics and so on. As such, it is important to have a grasp of how we codify a physical problem; we introduce this with an example:

Proposition 1.1 (Newton's Law of Cooling)**.** *If a body of temperature $T(t)$ is placed in an environment of temperature T_0 then it will cool at a rate proportional to the difference in temperature.*

> **Definition 1.2.** A *dependent variable* is a variables considered as changing as a consequence of changes in other variables, which are called *independent variables.*

In the example of Newton's Law of Cooling, the dependent variable is the temperature T which *depends* upon the independent variable time, t. The standard (Leibniz) notation for differentiation then gives us these equivalent forms for Newton's Law:

$$\begin{aligned}\frac{\mathrm{d}T}{\mathrm{d}t} &\propto T - T_0\\ \frac{\mathrm{d}T}{\mathrm{d}t} &= -k\left(T - T_0\right)\end{aligned}$$

where we take k to be a constant; in fact, we require the constant of proportionality $k > 0$ for actual physical temperature exchanges.

Having established this basic approach, we shall begin with a fairly informal overview of differentiation and integration, to help us understand the techniques we will develop later. For a fully rigorous (axiomatic) approach to calculus, see the Analysis courses.

1.1 Differentiation using Big O and Little-o Notation

We define the rate of change of a function $f(x)$ as being

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

which is pictorially equivalent to the gradient of f at x.

Note that the limit can be taken from above or below, written $\lim_{h\to 0^{\pm}} \frac{f(x+h)-f(x)}{h}$, with both side limits being equal for differentiable functions. (Hence $f(x) = |x|$ is not differentiable at $x = 0$.)

We use various notations, given $f = f(x)$:

$$\frac{\mathrm{d}f}{\mathrm{d}x} \equiv f'(x) \equiv \left(\frac{\mathrm{d}}{\mathrm{d}x}\right)[f(x)] \equiv \frac{\mathrm{d}}{\mathrm{d}x} f$$

where $\frac{\mathrm{d}}{\mathrm{d}x}$ is a *differential operator*. Then

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}f}{\mathrm{d}x}\right) \equiv \frac{\mathrm{d}^2 f}{\mathrm{d}x^2} \equiv f''(x)$$

To try and come up with a concise and useful way of writing f in terms of $\frac{\mathrm{d}f}{\mathrm{d}x}$, we introduce another notation (or two).

Definition 1.3. We write

$$f(x) = o(g(x))$$

as $x \to c$ if

$$\lim_{x\to c} \frac{f(x)}{g(x)} = 0$$

and we say f is *little-o* of g (as x tends to c).

This definition allows us to make explicit what we mean by f 'grows more slowly' than g.

Example 1.4.

(i) $x = o(\sqrt{x})$ as $x \to 0^+$.

(ii) $\ln x = o(x)$ as $x \to +\infty$.

Definition 1.5. We say that

$$f(x) = O(g(x))$$

as $x \to c$ if

$$\frac{f(x)}{g(x)}$$

is bounded as $x \to c$, and we say f is *big-O* of g (as x tends to c).

This similarly gives a rigorous definition of what it means to say that g 'grows at least as quickly' as f. Indeed, if $f = o(g)$ then it follows that $f = O(g)$.

Example 1.6. $\frac{2x^2-x}{x^2+1} = O(1)$ as $x \to \infty$. Similarly, $2x^2 - x = O\left(x^2+1\right) = O\left(x^2\right)$.

It follows that

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{f(x+h)-f(x)}{h} + \frac{o(h)}{h}$$

the term on the right being referred to as the *error term*. (If we wrote it as $\epsilon(h) = \frac{o(h)}{h}$ we would have a function ϵ such that $\frac{\epsilon}{h} \to 0$ as $h \to 0$.)

Thus

$$h\frac{\mathrm{d}f}{\mathrm{d}x} = f(x+h) - f(x) + o(h)$$

and hence

$$f(x+h) = \underbrace{f(x) + h\frac{\mathrm{d}f}{\mathrm{d}x}}_{\text{linear approximation}} + o(h)$$

Applying this to some fixed point x_0 we obtain, for a general point $x = x_0 + h$,

$$\begin{aligned} f(x_0 + h) &= f(x_0) + h\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right]_{x_0} + o(h) \\ f(x) &= f(x_0) + (x - x_0)\left[\frac{\mathrm{d}f}{\mathrm{d}x}\right]_{x_0} + o(h) \end{aligned}$$

Equivalently, if $y = f(x)$,

$$y = \underbrace{y_0 + m(x - x_0)}_{\text{linear approximation}} + o(h)$$

which gives the equation of the tangent line of the graph of $y = f(x)$ or $y(x)$.

1.1.1 The chain rule

Consider $f(x) = F[g(x)]$, assuming these are all differentiable functions in some interval. We use little-o notation to derive the *chain rule*, that $f'(x) = g'(x)F'[g(x)]$.

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0} \frac{F[g(x+h)] - F[g(x)]}{h} \\ &= \lim_{h\to 0} \frac{1}{h}\{F[g(x) + hg'(x) + o(h)] - F[g(x)]\} \\ &= \lim_{h\to 0} \frac{1}{h}\{F[g(x)] + [hg'(x) + o(h)]F'[g(x)] + o(hg'(x) + o(h)) - F[g(x)]\} \end{aligned}$$

Here note $o(h)F'[g(x)] = o(h)$ for finite F', and similarly $hg'(x)$ tends to 0 like h for finite g', so (noting that the $F[g(x)]$ terms cancel)

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} &= \lim_{h\to 0} \frac{1}{h}\{hg'(x)F'[g(x)] + o(h)\} \\ &= g'(x)F'[g(x)] \end{aligned}$$

This can be written as

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}f}{\mathrm{d}g} \cdot \frac{\mathrm{d}g}{\mathrm{d}x}$$

Example 1.7. If $f(x) = \sqrt{\sin x}$, then $g(x) = \sin x$ and $F[x] = \sqrt{x}$, so $f'(x) = \cos x \cdot \frac{1}{2\sqrt{\sin x}}$.

1.1.2 The inverse function rule

A special case of this is that

$$1 = \frac{\mathrm{d}x}{\mathrm{d}x} = \frac{\mathrm{d}x}{\mathrm{d}y} \cdot \frac{\mathrm{d}y}{\mathrm{d}x}$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1}$$

If we have $y = f(x)$, then $\mathrm{d}y/\mathrm{d}x = f'(x)$.

Now $x = f^{-1}(y)$ implies that

$$\frac{\mathrm{d}x}{\mathrm{d}y} = \left(f^{-1}\right)'(y)$$

but then from the above, we have

$$\begin{aligned}\left(f^{-1}\right)'(y) &= \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^{-1} \\ &= \left(f'(x)\right)^{-1}\end{aligned}$$

Rewriting everything in terms of one variable,

$$\left(f^{-1}\right)'(y) = \frac{1}{f'\left(f^{-1}(y)\right)}$$

1.1.3 The product rule

We can also show that if $f(x) = u(x)v(x)$, where u and v are differentiable, $\frac{\mathrm{d}f}{\mathrm{d}x} = u'v + uv'$, using a similar argument.

Exercise 1.8. Prove the product rule using little-o notation.

1.1.4 Leibniz's rule

If $f = uv$ we have

$$\begin{aligned}f &= uv \\ f' &= u'v + vu' \\ f'' &= u''v + u'v' + v'u' + vu'' \\ &= u''v + 2u'v' + uv'' \\ f''' &= u'''v + 3''v' + 3u'v'' + 3v'''\end{aligned}$$

This suggests a pattern, with the numbers echoing Pascal's triangle. Indeed, it is fairly easy to show (by induction) that

$$f^{(n)}(x) = u^{(n)}v + nu^{(n-1)}v' + \cdots + \binom{n}{r}u^{(n-r)}v^{(r)} + \cdots + uv^{(n)}$$

This is referred to as *(the general) Leibniz's rule.*

1.1.5 Taylor series

Recall that

$$f(x+h) = f(x) + hf'(x) + o(h)$$

Now imagine that $f'(x)$ is differentiable, and so on; then we can construct

$$\begin{aligned} f(x+h) &= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + o\left(h^2\right) \\ &= f(x) + hf'(x) + \frac{h^2}{2!} f''(x) + \cdots + \frac{h^n}{n!} f^{(n)}(x) + E_n \end{aligned}$$

where $E_n = o(h^n)$ as $h \to 0$, provided $f^{(n+1)}(x)$ exists.

Theorem 1.9 (Taylor's Theorem)**.** *Taylor's theorem gives us various ways of writing E_n, but the most useful consequence is that $E_n = O\left(h^{n+1}\right)$ as $h = x - x_0 \to 0$, where*

$$f(x) = f(x_0) + (x - x_0) f'(x_0) + \cdots + \frac{(x - x_0)^n}{n!} f^{(n)}(x_0) + E_n$$

This leads to the Taylor series about x_0 and gives a useful *local* approximation for f. We cannot expect it to be of much use when $x - x_0$ grows large.

1.1.6 L'Hôpital's rule

Let $f(x)$ and $g(x)$ be differentiable at $x = x_0$, and

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0$$

Then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

provided $g'(x_0) \neq 0$ (and that the limit exists).

Proof. From Taylor series, we have

$$\begin{aligned} f(x) &= f(x_0) + (x - x_0) f'(x_0) + o(x - x_0) \\ g(x) &= g(x_0) + (x - x_0) g'(x_0) + o(x - x_0) \end{aligned}$$

and hence $f(x_0) = g(x_0) = 0$; then

$$\frac{f}{g} = \frac{f' + \frac{o(\Delta x)}{\Delta x}}{g' + \frac{o(\Delta x)}{\Delta x}}$$

where $\Delta x = x - x_0$. Then as $\Delta x \to 0$,

$$\frac{f}{g} \to \frac{f'}{g'}$$

□

1.2 Integration

The basic idea behind integration is of a refined sum of the area under a graph; for example, a simple definition of the Riemann integral, where it exists, might be given by

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{\Delta x \to 0} \sum_{n=0}^{N-1} f(x_n)\,\Delta x$$

where x_n are N points spaced throughout $[a, b]$, where $\Delta x = x_n - x_{n-1}$ is the size of the interval being summed over.

Note that if f' is finite, the difference between the area beneath the graph and the approximation given above is a roughly triangular shape of area $|E_n|$, where the base is Δx and the height is $f'(x_n)\,\Delta x + o(\Delta x)$. Hence

$$\begin{aligned} E_n &= \frac{1}{2}\Delta x\left[f'(x_n)\,\Delta x + o(\Delta x)\right] \\ &= O\left((\Delta x)^2\right) \end{aligned}$$

for finite f'.

Hence since we choose x_n above so that the $\Delta x = O\left(\frac{1}{N}\right)$, we get

$$\begin{aligned} \text{Area} &= \lim_{N\to\infty}\left[\sum_{n=1}^{N-1} f(x_n)\,\Delta x + O\left(\frac{1}{N}\right)\right] \\ &= \lim_{N\to\infty}\left[\sum_{n=1}^{N-1} f(x_n)\,\Delta x\right] \\ &= \int_a^b f(x)\,\mathrm{d}x \end{aligned}$$

as defined above.

1.2.1 The Fundamental Theorem of Calculus

We now understand intuitively what we mean by integration, but we need to develop the relationship between the function f and its integral. It is traditional to say that integration is the opposite of differentiation. The following theorem says exactly what we mean by this.

Theorem 1.10 (The Fundamental Theorem of Calculus)**.** *If f is continuous, $\frac{\mathrm{d}}{\mathrm{d}x}\int_a^x f(t)\,\mathrm{d}t = f(x)$.*

Proof. We shall not claim to give a full proof of the above theorem, but if we assume that if $\int_x^{x+h} f(t)\,\mathrm{d}t = f(x)\,h + O\left(h^2\right)$ we can establish it. The partial proof is obtained simply be setting

$F(x) = \int_a^x f(t)\,\mathrm{d}t$ and calculating

$$\begin{aligned}
\frac{\mathrm{d}F}{\mathrm{d}x} &= \lim_{h\to 0}\frac{1}{h}\left[\int_a^{x+h} f(t)\,\mathrm{d}t - \int_a^x f(t)\,\mathrm{d}t\right] \\
&= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,\mathrm{d}t \\
&= \lim_{h\to 0}\frac{1}{h}\left[f(x)\,h + O\left(h^2\right)\right] \\
&= f(x)
\end{aligned}$$

□

This means that if we have a continuous function f that we recognize as the derivative of some other function F, we can infer that $\int_a^x f(t)\,\mathrm{d}t = F(x) - F(a)$.

Remark. Note that we are taking the derivative with respect to the upper limit; the corresponding result for the lower limit is

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_x^b f(t)\,\mathrm{d}t = -f(x)$$

as can be seen by exchanging the two limits.

Additionally, we can establish, via the chain rule, that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\int_a^{g(x)} f(t)\,\mathrm{d}t &= \frac{\mathrm{d}g}{\mathrm{d}x}\frac{\mathrm{d}}{\mathrm{d}g}\int_a^g f(t)\,\mathrm{d}t \\
&= g'(x)\cdot f(g(x))
\end{aligned}$$

Finally, we can define the *indefinite integral* for convenience:

Definition 1.11. A *indefinite integral* is written as

$$\int f(x)\,\mathrm{d}x \equiv \int^x f(t)\,\mathrm{d}t$$

where the lower limit is unspecified, giving rise to the familiar arbitrary additive constant.

1.2.2 Integration by substitution

Now using the fundamental theorem of calculus, we can reverse our rules for differentiation to obtain techniques for integration. In the case of the chain rule, this gives rise to the familiar method of integration by substitution:

$$\int f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int f(u)\,\mathrm{d}u$$

or more accurately, giving explicit limits,

$$\int_a^b f(u(x))\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x = \int_{u(a)}^{u(b)} f(t)\,\mathrm{d}t$$

Note that we are again assuming sufficient continuity of f, and also of u (since we need it to have a derivative). It is then clear that if $F = \int f$ is an antiderivative of f, then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}F\left(u\left(x\right)\right) &= u'\left(x\right)F'\left(u\left(x\right)\right)\\ &= u'\left(x\right)f\left(u\left(x\right)\right)\end{aligned}$$

and hence that

$$\begin{aligned}\int_a^b u'\left(x\right)f\left(u\left(x\right)\right)\mathrm{d}x &= F\left(u\left(b\right)\right) - F\left(u\left(a\right)\right)\\ &= \int_{u(a)}^{u(b)} f\left(t\right)\mathrm{d}t\end{aligned}$$

as was required.

Example 1.12. For completeness, we give an example:

$$\begin{aligned}I &= \int \frac{1-2x}{\sqrt{x-x^2}}\mathrm{d}x\\ &= \int \left(x-x^2\right)^{-1/2}\left(1-2x\right)\mathrm{d}x\end{aligned}$$

Then $u = x - x^2$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = (1-2x)$, so the integral becomes

$$\begin{aligned}I &= \int u^{-1/2}\mathrm{d}u\\ &= 2u^{1/2} + C\\ &= 2\sqrt{x-x^2} + C\end{aligned}$$

Note that we avoided writing $\mathrm{d}u = (1-2x)\,\mathrm{d}x$, since we have not really given a meaning to this expression. This is best used only as a mnemonic.

Integration by substitution often requires a certain amount of inventiveness, so it is useful to have some rules of thumb for dealing with some forms of integrand. This table has some suggestions for trigonometric substitutions:

Identity	Function	Substitution
$\cos^2\theta + \sin^2\theta = 1$	$\sqrt{1-x^2}$	$x = \sin\theta$
$1 + \tan^2\theta = \sec^2\theta$	$1 + x^2$	$x = \tan\theta$
$\cosh^2 u - \sinh^2 u = 1$	$\sqrt{x^2-1}$ $\sqrt{x^2+1}$	$x = \cosh u$ $x = \sinh u$
$1 - \tanh^2 u = \mathrm{sech}^2 u$	$1 - x^2$	$x = \tanh u$

Example 1.13. This gives one example of the application of such an identity: consider $\int \sqrt{2x - x^2}\mathrm{d}x$. An often useful approach with expressions like this is to complete the square:

$$\int \sqrt{2x - x^2}\mathrm{d}x = \int \sqrt{1 - (x-1)^2}\mathrm{d}x$$

Then this has the form above suggesting a $\sin\theta$ substitution. In this case, we want $x - 1 = \sin\theta$, and so we have

$$\begin{aligned} \int \sqrt{2x - x^2}\mathrm{d}x &= \int \sqrt{1 - (\sin\theta)^2}\frac{\mathrm{d}x}{\mathrm{d}\theta}\mathrm{d}\theta \\ &= \int \cos\theta \cdot \cos\theta \mathrm{d}\theta \\ &= \int \cos^2\theta \mathrm{d}\theta \end{aligned}$$

We can then calculate this using the standard double-angle formula $\cos 2\theta = 2\cos^2\theta - 1$:

$$\begin{aligned} \int \cos^2\theta \mathrm{d}\theta &= \int \frac{1}{2}(1 + \cos 2\theta)\,\mathrm{d}\theta \\ &= \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C \\ &= \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C \\ &= \frac{1}{2}\sin^{-1}(x-1) + \frac{1}{2}(x-1)\sqrt{1 - (x-1)^2} + C \end{aligned}$$

Another example shows that choosing the right substitution may not be obvious:

Example 1.14. Calculate $\int \frac{x}{1+x^4}\mathrm{d}x$.

It might seem tempting to try and simplify the denominator in this expression, but the best approach in fact is to avoid having terms both in the numerator *and* denominator, which is most easily done by eliminating the x term on top. Here, let $u = x^2$ gives $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and thus we get

$$\int \frac{x}{1+x^4}\mathrm{d}x = \frac{1}{2}\int \frac{1}{1+u^2}\mathrm{d}u$$

Then using the $u = \tan\theta$ substitution, we get

$$\begin{aligned} \frac{1}{2}\int \frac{1}{1+u^2}\mathrm{d}u &= \frac{1}{2}\int \frac{1}{1+\tan^2\theta}\sec^2\theta \mathrm{d}\theta \\ &= \frac{1}{2}\theta + C \\ &= \frac{1}{2}\tan^{-1}x^2 + D \end{aligned}$$

1.2.3 Integration by parts

Again, we can use the Leibniz identity for differentiation, $(uv)' = u'v + uv'$, to obtain an integration technique:

$$\begin{aligned} \int (u'v + uv')\,\mathrm{d}x &= uv \\ \int uv'\mathrm{d}x &= uv - \int u'v\mathrm{d}x \end{aligned}$$

Again, this demands some inventiveness in its application - identifying what to call u and what to call v' is not always easy.

We begin with a straight-forward example:

Example 1.15. Calculate $\int_0^\infty xe^{-x}\mathrm{d}x$.

To simplify the integral we want to make the $u'v$ expression easier to integrate. Taking $u = x$ and $v' = e^{-x}$ gives

$$\begin{aligned} \int_0^\infty xe^{-x}\mathrm{d}x &= \left[-xe^{-x}\right]_0^\infty - \int_0^\infty -e^{-x}\mathrm{d}x \\ &= 0 + \left[-e^{-x}\right]_0^\infty \\ &= 0 + 1 \\ &= 1 \end{aligned}$$

This technique but can actually give surprising results:.

Example 1.16. Calculate $\int e^x \cos x\mathrm{d}x$.

Using the technique, we write $u = \cos x$ and $v' = e^x$, which gives us

$$\int e^x \cos x\mathrm{d}x = e^x \cos x + \int e^x \sin x\mathrm{d}x$$

with a similar integral to be computed. We can now take $u = \sin x$, and $v' = e^x$ again, to get

$$\int e^x \sin x\mathrm{d}x = e^x \sin x - \int e^x \cos x\mathrm{d}x$$

Hence we have

$$\begin{aligned} \int e^x \cos x\mathrm{d}x &= e^x \cos x + e^x \sin x - \int e^x \cos x\mathrm{d}x \\ 2\int e^x \cos x\mathrm{d}x &= e^x\left[\sin x + \cos x\right] + C \\ \int e^x \cos x\mathrm{d}x &= \frac{1}{2}e^x\left[\sin x + \cos x\right] + D \end{aligned}$$

Remark. Another approach to this is to use complex numbers: we remember Euler's identity, $e^{ix} = \cos x + i \sin x$, and then see

$$\int e^{(1+i)x} \mathrm{d}x = \int e^x \cos x \mathrm{d}x + i \int e^x \sin x \mathrm{d}x$$

Taking real parts, we then get

$$\begin{aligned}
\mathrm{Re}\left\{\frac{1}{1+i} e^{(1+i)x}\right\} + C &= \int e^x \cos x \mathrm{d}x \\
\int e^x \cos x \mathrm{d}x &= \mathrm{Re}\left\{\frac{1-i}{2} e^{ix}\right\} e^x + C \\
&= \frac{1}{2} e^x \left[\cos x + \sin x\right] + C
\end{aligned}$$

Exercise 1.17. Calculate $\int \sec^3 x \mathrm{d}x$.

A particularly common application of integration by parts is in proving *reduction formulae.*

Example 1.18. Calculate $I_n = \int \cos^n x \mathrm{d}x$ in terms of I_{n-2}.

We can do this by setting $u = \cos x$ and $v' = \cos^{n-1} x$:

$$\begin{aligned}
I_n &= \cos^{n-1} x \sin x + \int (n-1) \cos^{n-2} x \sin^2 x \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x \left(1 - \cos^2 x\right) \mathrm{d}x \\
&= \cos^{n-1} x \sin x + (n-1) \left(I_{n-2} - I_n\right) \\
nI_n &= \cos^{n-1} x \sin x + (n-1) I_{n-2} \\
I_n &= \frac{1}{n} \cos^{n-1} x \sin x + \frac{n-1}{n} I_{n-2}
\end{aligned}$$

where we ignore the various integration constants.

2 Functions of Several Variables

One of the most natural extensions of this kind of calculus is to generalize from $f(x)$ to $f(x, y, \cdots, z)$, or $f(\mathbf{x})$ for a vector $\mathbf{x}$. These functions obviously naturally arise in all sorts of situations, including any physical situation with a function defined on all space, like electromagnetic fields.

To begin with, we will focus on functions of two variables $f(x, y)$. Examples abound, with one of the most useful being the height of terrain, $z(x, y)$; pressure maps at sea level also provide a familiar example. However, we will also consider more abstract parameterizations, like the density of a gas as a function of temperature and pressure, $\rho(T, p)$.

These functions are best represented by contour plots in 2D (or surface/heightmap plots in simulated 3D), and we will see how to draw these diagrams, as well as how to analyze them.

What is the slope of a hill?

This is natural motivating question for differential calculus on surfaces. The key observation is that *slope depends on direction.* We begin by thinking about the slopes in directions parallel to the coordinate axes.

Definition 2.1. The *partial derivative* of $f(x, y)$ with respect to x is the rate of change of f with respect to x keeping y constant. We write

$$\left.\frac{\partial f}{\partial x}\right|_y = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

The definition of $\partial f/\partial y|_x$ is similar. We shall often omit explicitly writing the variable(s) we hold constant, where there is minimal danger or confusion - in general, if $f = f(x_1, x_2, \cdots, x_n)$ then we assume $\partial f/\partial x_i$ means holding *all* x_j constant for $j \neq i$. Note that it makes no sense to take a derivative with respect to a variable held constant!

It is important to note that the partial derivative is still a function of both x and y, like f. We denote the partial derivative with respect to x at (x_0, y_0) by

$$\left.\frac{\partial f}{\partial x}\right|_{x=x_0, y=y_0} \text{ or } \frac{\partial f}{\partial x}(x_0, y_0) \text{ or } \left.\frac{\partial f}{\partial x}\right|_y (x_0, y_0)$$

depending on what looks most readable. The unfortunate ambiguity in the appearance of these notations is not usually too much of a problem.

Example 2.2. For example, if

$$f(x, y) = x^2 + y^3 + e^{xy^2}$$

then we have

$$\left.\frac{\partial f}{\partial x}\right|_y = 2x + y^2 e^{xy^2}$$
$$\left.\frac{\partial f}{\partial y}\right|_x = 3y^2 + 2xye^{xy^2}$$

Leaving out the explicit subscripts, and extending Leibniz's notation to partial derivatives, we also get

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= 2 + y^4 e^{xy^2} \\
\frac{\partial^2 f}{\partial y^2} &= 6y + 2xe^{xy^2} + 4x^2y^2e^{xy^2} \\
\frac{\partial^2 f}{\partial x \partial y} &\equiv \frac{\partial}{\partial x}\left[\frac{\partial f}{\partial y}\right] \\
&= 2ye^{xy^2} + 2xy^3e^{xy^2} \\
\frac{\partial^2 f}{\partial y \partial x} &\equiv \frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}\right] \\
&= 2ye^{xy^2} + 2xy^3e^{xy^2}
\end{aligned}$$

Note that the so-called *mixed* second partial derivatives mean

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \left.\frac{\partial}{\partial x}\right|_y \left.\frac{\partial f}{\partial y}\right|_x$$

which is very cumbersome to write out.

It is actually a general rule that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

so long as we are working in flat space, and all of the second partial derivatives are continuous - in general, it is called *Clairaut's theorem* or *Schwarz's theorem:*

Theorem 2.3 (Clairaut). *If a real-valued function $f(x_1, x_2, \cdots, x_n)$ with n real parameters x_i (so f maps $\mathbb{R}^n \to \mathbb{R}$) has continuous second partial derivatives at $(a_1, a_2, \cdots, a_n)$ then*

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$$

at that point.

We will not prove this theorem here, but we usually assume that this is valid without worrying about the details.

Note that while we will often neglect to indicate which variables are fixed, assuming all those not mentioned are, it does make a difference.

Example 2.4. If $f(x,y,z) = xyz$ then

$$\begin{aligned}\frac{\partial f}{\partial x} &\equiv \left.\frac{\partial f}{\partial x}\right|_{y,z} \\ &= yz\end{aligned}$$

but

$$\left.\frac{\partial f}{\partial x}\right|_{y} = y\left(x\frac{\partial z}{\partial x} + z\right)$$

This assumes that we have some idea of z varying with x - typically, we might look at the values of f on a surface in 3 dimensional space, so we could have $z = z(x,y)$ on that surface.

Remark. Another very useful notation for partial derivatives is to write

$$f_x \equiv \frac{\partial f}{\partial x}$$

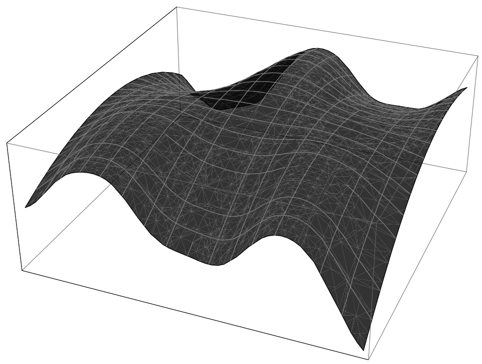
and then for higher derivatives,

$$\begin{aligned}f_{\boxed{xy}} &\equiv \frac{\partial^2 f}{\boxed{\partial y \partial x}} \\ &\equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)\end{aligned}$$

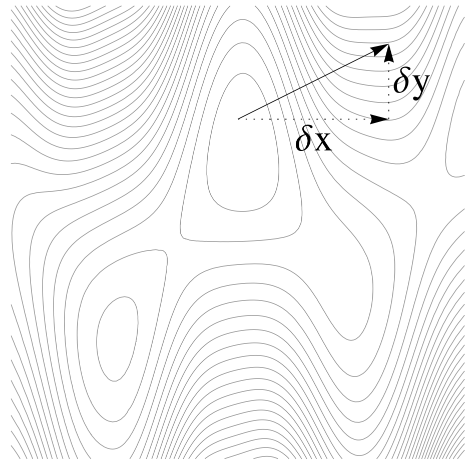
Note that the order of the subscripts is the order that the derivatives are taken in, not the order they are written in, in Leibniz's notation.

We will not make much use of this notation in this course, preferring to keep subscripts for indexing.

2.1 The Chain Rule



(a) Surface



(b) Contours and a short path

Figure 2.1: An example of a family

Imagine we are walking across a hilly landscape, like that shown in Figure 2.1, where the height at any point is given by $z = f(x, y)$. Say we walk a small distance δx in the x direction, and *then* a small distance δy in the y direction. Then the total change in height is

$$\begin{aligned} \delta f &= f(x+\delta x, y+\delta y) - f(x, y) \\ &= [f(x+\delta x, y+\delta y) - f(x+\delta x, y)] \\ &\quad + [f(x+\delta x, y) - f(x, y)] \end{aligned}$$

But then, if we assume f is reasonably smooth in both the x and y directions around the point (x, y), we can deduce that

$$\begin{aligned} \delta f &= \frac{\partial f}{\partial y}(x+\delta x, y) \cdot \delta y + o(\delta y) \\ &\quad + \frac{\partial f}{\partial x}(x, y) \cdot \delta x + o(\delta x) \\ &= \frac{\partial f}{\partial y}(x, y) \cdot \delta y + o(\delta x) \cdot \delta y + o(\delta y) \\ &\quad + \frac{\partial f}{\partial x}(x, y) \cdot \delta x + o(\delta x) \\ &= \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial x}\delta x + o(\delta x, \delta y) \end{aligned}$$

where the term $o(\delta x, \delta y)$ is taken as meaning a collection of terms that tend to 0 faster than either δx or δy when δx and δy tend to 0.

Informally, when we take the limit $\delta x, \delta y \to 0$ we get an expression for an 'infinitesimal' change in

f, given infinitesimal changes in x and y, which can be written as

$$\boxed{\mathrm{d}f = \frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial x}\mathrm{d}x}$$

which is called the *chain rule in differential form* (terms with upright 'd's, like $\mathrm{d}f$, are differentials).

We write this on the understanding that we will either sum the differentials, as in

$$\int \cdots \mathrm{d}f = \int \cdots \left(\frac{\partial f}{\partial y}\mathrm{d}y + \frac{\partial f}{\partial x}\mathrm{d}x\right)$$

or divide by another infinitesimal, as in

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}}$$

or

$$\boxed{\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t}}$$

before taking the limit. These last forms are technically the correct way of stating the chain rule.

The first is best viewed as saying that, if we walk along a path $x = x(t)$ and $y = y(t)$ as time t passes, then the rate at which f changes along the path $(x(t), y(t))$ is given by the above formula for $\mathrm{d}f/\mathrm{d}t$. The second form assumes that we have $f = f(x(t,u), y(t,u))$ so that our path depends on two variables t and u; we can then look at the rate of change of f along the path as we vary *one* of these parameters. For example, we might have a family of paths indexed by u, with each path being parameterized by time t. We can then ask, given that we stick to the path given by u_0, what is the rate of change $\partial f/\partial t$ along this path?

The first formula can be derived from the original expression in terms of $o(\delta x, \delta y)$ by taking the following limit with $f = f(x(t), y(t))$:

$$\begin{aligned}\frac{\mathrm{d}f}{\mathrm{d}t} &= \lim_{\delta t \to 0}\frac{\delta f}{\delta t} \\ &= \lim_{\delta t \to 0}\left[\frac{\partial f}{\partial x}\frac{\delta x}{\delta t} + \frac{\partial f}{\partial y}\frac{\delta y}{\delta t} + \frac{o(\delta x, \delta y)}{\delta t}\right]\end{aligned}$$

Then *provided* $\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t}$ *is finite*, we have $\delta x \approxeq \dot{x}\delta t$ and so if $\epsilon = o(\delta x)$ then $\epsilon = o(\delta t)$. From this it follows that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}$$

as required.

A special case arises from specifying a path by $y = y(x)$. Then, along this path, we have

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}x}$$

Remark. If we add more variables, the chain rule is extended simply by adding more terms to the sum.

So if $f = f(x_1, x_2, \cdots, x_n)$ we would discover, for example, that

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial f}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} + \cdots + \frac{\partial f}{\partial x_n}\frac{\mathrm{d}x_n}{\mathrm{d}t}$$

One very useful way of looking at this expression in general is to look at the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, and to realize that if we had some other vector

$$\mathbf{v} = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_n}\right)$$

then we could simply write the chain rule as a dot product, making it look much more like the original chain rule:

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \mathbf{v} \cdot \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t}$$

This is exactly what we do in the next section.

2.2 Two-Dimensional Calculus

In this section, we see how some key ideas to do with differentiation from basic calculus carry over to a higher dimensional case (though we only work with two dimensions for simplicity).

2.2.1 Directional derivatives

Consider a vector displacement $\mathrm{d}\mathbf{s} = (\mathrm{d}x, \mathrm{d}y)$ like that shown as the arrow in Figure 2.1. The change in $f(x, y)$ over that displacement is

$$\begin{aligned} \mathrm{d}f &= \frac{\partial f}{\partial x}\mathrm{d}x + \frac{\partial f}{\partial y}\mathrm{d}y \\ &= (\mathrm{d}x, \mathrm{d}y) \cdot \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) \\ &= \mathrm{d}\mathbf{s} \cdot \boldsymbol{\nabla} f \end{aligned}$$

where we define $\boldsymbol{\nabla} f$ (which we read 'grad f' or 'del f' - the symbol is a *nabla*) to satisfy this relationship:

> **Definition 2.5.** In two dimensional Cartesian coordinates,
>
> $$\mathrm{grad} f \equiv \boldsymbol{\nabla} f \equiv \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

Remark. Note that $(\boldsymbol{\nabla} f)(x, y)$ is a function of position, or a *field* - since $\boldsymbol{\nabla} f$ is a vector, it is a *vector field.*

Note that if we write $\mathrm{d}\mathbf{s} = \hat{\mathbf{s}}\mathrm{d}s$ where $|\hat{\mathbf{s}}| = 1$ so that $\hat{\mathbf{s}}$ is a unit vector, we have

$$\boxed{\frac{\mathrm{d}f}{\mathrm{d}s} = \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f}$$

which is the *directional derivative of* f in the direction of $\hat{\mathbf{s}}$.

This gives us the striking relationship

$$\left|\frac{\mathrm{d}f}{\mathrm{d}s}\right| = |\boldsymbol{\nabla} f| \cos\theta$$

so that the rate of change of f at a point varies exactly like $\cos\theta$ as we look in different directions at angle θ. We can make a few observations based on this:

(i) $\max_\theta \frac{\mathrm{d}f}{\mathrm{d}s} = |\boldsymbol{\nabla} f|$ so $\boldsymbol{\nabla} f$ always has a magnitude equal to maximum rate of change of $f(x, y)$ in any direction.

(ii) The direction of $\boldsymbol{\nabla} f$ is the direction in which f increases most rapidly - hence '$\boldsymbol{\nabla} f$ always points uphill'.

(iii) If $d\mathbf{s}$ is a displacement along a contour of f, then by definition

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}s} = 0 \quad &\iff \quad \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f = 0 \\ &\iff \quad \boldsymbol{\nabla} f \text{ is orthogonal to the contour} \end{aligned}$$

so contours correspond exactly to lines at right-angles to the gradient $\boldsymbol{\nabla} f$.

2.2.2 Stationary points

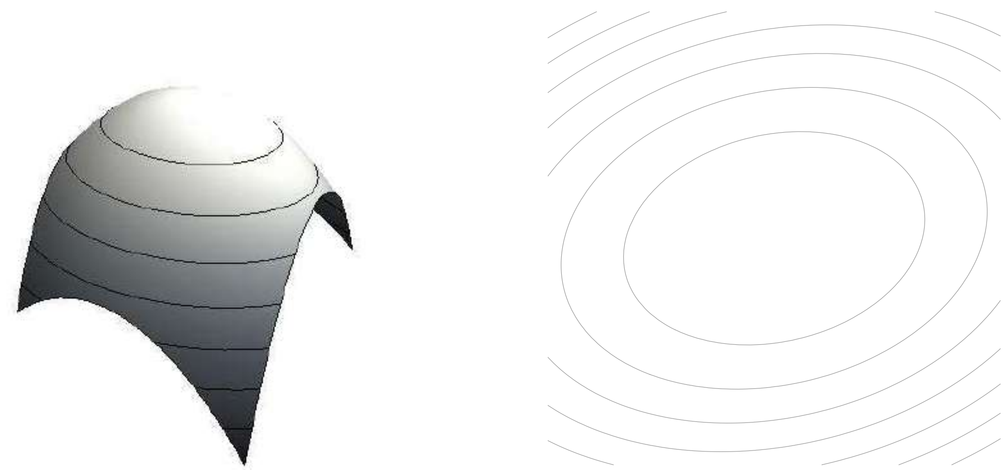
The last point we noted above tells us that there is always one direction in which $\frac{\mathrm{d}f}{\mathrm{d}s}$ is zero - but what about true maxima and minima?

Local maxima and minima must have $\frac{\mathrm{d}f}{\mathrm{d}s} = 0$ in *all* directions:

$$\begin{aligned} \hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f &= 0 \quad \forall \hat{\mathbf{s}} \\ \iff \boldsymbol{\nabla} f &= \mathbf{0} \\ \iff \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} &= 0 \end{aligned}$$

This is probably not a very surprising result, but like the idea of a one-dimensional stationary point, it is fundamental.

The cases of a local maximum and a local minimum are displayed in Figures 2.2 and 2.3 respectively, along with the contour plots for the shown surfaces. Note that at a two-dimensional local maximum, the function is at a maximum in both the x and y directions, and vice versa - similarly for minima. However, we get an interesting new case when the function is a maximum along one direction, and a minimum along another. This case is shown in Figure 2.4, and is one example of a s-called *saddle point* - this generalizes the idea of the one-dimensional case of a curve which is stationary without being an extremum of any kind (like at $x = 0$ for the curve $y = x^3$).

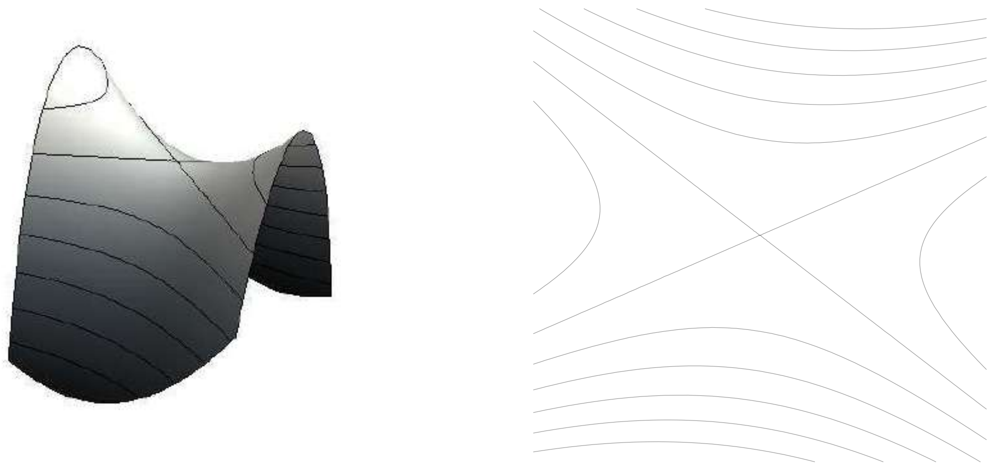


(a) Surface (b) Contours

Figure 2.2: An example of a local maximum

(a) Surface (b) Contours

Figure 2.3: An example of a local minimum



(a) Surface (b) Contours

Figure 2.4: An example of a saddle point

2.2.3 Multidimensional Taylor series

Once we have developed ideas of directional differentiation, and looked at finding stationary points, it is natural to try to extend the idea of using the second derivative to classify these points as shown above. Since we formally deduce the nature of stationary points in the one-dimensional case by looking at the second-order terms in the local Taylor series for the function about the point, it will be useful to have an expression analogous to

$$f(x) = f(a) + (x-a) f'(a) + \frac{1}{2}(x-a)^2 f''(a) + \cdots$$

Therefore, we will now deduce an expression for the Taylor series of a function $f(\mathbf{x})$ for a vector $\mathbf{x}$ (thinking mainly of the two-dimensional case, though the same process extends easily to higher dimensions). Then, in section 2.2.4, we will discover how to actually classify these points.

Since we have no techniques for dealing with vector series yet, it is natural to work in the case we already know about: the behaviour of $f(\mathbf{x})$ along an arbitrary line $\mathbf{x}_0 + s\hat{\mathbf{s}}$ through the point x_0. Consider a small finite displacement $\delta\mathbf{s} = (\delta s)\hat{\mathbf{s}}$ along this line. Then the Taylor series for $f(\mathbf{x}) = f(\mathbf{x}(s))$ along this line is given by

$$\begin{aligned} f(\mathbf{x}) = f(\mathbf{x}_0 + \delta\mathbf{s}) &= f(\mathbf{x}_0) + \delta s \frac{\mathrm{d}f}{\mathrm{d}s} + \frac{1}{2}(\delta s)^2 \frac{\mathrm{d}^2 f}{\mathrm{d}s^2} + \cdots \\ &= f(\mathbf{x}_0) + (\delta s)\,\hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f + (\delta s)^2 \cdot (\hat{\mathbf{s}} \cdot \boldsymbol{\nabla})(\hat{\mathbf{s}} \cdot \boldsymbol{\nabla}) f + \cdots \\ &= f(\mathbf{x}_0) + (\delta s)\,\hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f + (\delta s)^2 \cdot (\hat{\mathbf{s}} \cdot \boldsymbol{\nabla})^2 f + \cdots \end{aligned}$$

where

$$(\delta s)\,\hat{\mathbf{s}} \cdot \boldsymbol{\nabla} f = \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y}$$

and

$$\begin{aligned}(\delta s)^2 \cdot (\hat{\mathbf{s}} \cdot \boldsymbol{\nabla})^2 f &= (\delta s)^2 \cdot \left[\hat{s}_x \frac{\partial}{\partial x} + \hat{s}_y \frac{\partial}{\partial y}\right]^2 f \\ &= (\delta s)^2 \cdot \left[\hat{s}_x^2 \frac{\partial^2 f}{\partial x^2} + \hat{s}_x \hat{s}_y \frac{\partial^2 f}{\partial x \partial y} + \hat{s}_y \hat{s}_x \frac{\partial^2 f}{\partial y \partial x} + \hat{s}_y^2 \frac{\partial^2 f}{\partial y^2}\right] \\ &= (\delta x)^2 f_{xx} + (\delta x)(\delta y) f_{yx} + (\delta y)(\delta x) f_{xy} + (\delta y)^2 f_{yy}\end{aligned}$$

recalling that

$$\boldsymbol{\nabla} \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$$

Remark. Note that the power of two after the *operator* $[\hat{s}_x \partial/\partial x + \hat{s}_y \partial/\partial y]$ indicates that it is applied twice, *not* that the result is squared.

To simplify this expression, it is natural to take advantage of the new vector notation we have adopted, and write this last term as follows:

$$\begin{aligned}(\delta s)^2 \cdot (\hat{\mathbf{s}} \cdot \boldsymbol{\nabla})^2 f &\equiv \begin{pmatrix}\delta x & \delta y\end{pmatrix} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix} \begin{pmatrix}\delta x \\ \delta y\end{pmatrix} \\ &\equiv \delta \mathbf{s} \cdot (\boldsymbol{\nabla}\boldsymbol{\nabla} f) \cdot \delta \mathbf{s}\end{aligned}$$

Definition 2.6. The matrix

$$\boldsymbol{\nabla}\boldsymbol{\nabla} f = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy}\end{pmatrix}$$

is called the *Hessian matrix* of the function f.

Remark. We do *not* write $\boldsymbol{\nabla}^2 f$ for this matrix because this notation has a special meaning; it is the *Laplacian* $\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f)$ which is a scalar, not a matrix.

Thus in coordinate-free form we have the Taylor series

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \delta \mathbf{x} \cdot \boldsymbol{\nabla} f(\mathbf{x}_0) + \frac{1}{2} \delta \mathbf{x} \cdot [\boldsymbol{\nabla}\boldsymbol{\nabla} f] \cdot \delta \mathbf{x} + \cdots$$

For the case of two-dimensional Cartesian coordinates, with continuous second partial derivatives, we can write

$$\begin{aligned}f(x,y) &= f(x_0,y_0) + (x-x_0) f_x + (y-y_0) f_y \\ &\quad + \frac{1}{2}\left[(x-x_0)^2 f_{xx} + 2(x-x_0)(y-y_0) f_{xy} + (y-y_0)^2 f_{yy}\right] \\ &\quad + \cdots\end{aligned}$$

With this expression, we can move on to work out the behaviour of f near a stationary point.

2.2.4 Classification of stationary points

Obviously, near a stationary point, by definition $\nabla f = 0$. Therefore, the behaviour of f is given by

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \frac{1}{2}\delta\mathbf{x}\cdot\mathbf{H}\cdot\delta\mathbf{x} + \cdots$$

where we have written $\mathbf{H} = \nabla\nabla f$ for the Hessian.

In this section, we will not consider the cases where second derivatives vanish - therefore, the properties of the matrix $\mathbf{H}$ are all that matter.

At a minimum, we must have $f(\mathbf{x}) > f(\mathbf{x}_0)$ for all sufficiently small $\delta\mathbf{x}$, regardless of which direction $\delta\mathbf{x}$ points in - that implies that

$$\delta\mathbf{x}\cdot\mathbf{H}\cdot\delta\mathbf{x} > 0 \quad \forall\delta\mathbf{x} \neq \mathbf{0}$$

noting that the magnitude of $\delta\mathbf{x}$ is irrelevant to the sign of the result. Similarly, for a maximum

$$\delta\mathbf{x}\cdot\mathbf{H}\cdot\delta\mathbf{x} < 0 \quad \forall\delta\mathbf{x} \neq \mathbf{0}$$

and at a saddle point, $\delta\mathbf{x}\cdot\mathbf{H}\cdot\delta\mathbf{x}$ takes both signs.

These properties of the matrix are given names:

Definition 2.7. If $\mathbf{v}^\dagger\mathbf{A}\mathbf{v} = \mathbf{v}\cdot(\mathbf{A}\mathbf{v}) > 0$ for all vectors $\mathbf{v}$ not equal to the zero vector, then the matrix $\mathbf{A}$ is said to be *positive definite*. We sometimes write $\mathbf{A} > 0$.

If $\mathbf{v}^\dagger\mathbf{A}\mathbf{v} < 0$ for all $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{A}$ is said to be *negative definite*. Similarly, we sometimes write $\mathbf{A} < 0$.

An *indefinite* matrix $\mathbf{A}$ is one for which vectors $\mathbf{v}$ and $\mathbf{w}$ exist with

$$\mathbf{v}^\dagger\mathbf{A}\mathbf{v} > 0 > \mathbf{w}^\dagger\mathbf{A}\mathbf{w}$$

In order to understand how we can find out whether a matrix is positive definite etc., it is helpful to first consider the case $f_{xy} = f_{yx} = 0$, so that the matrix $\mathbf{A}$ is diagonal:

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} f_{xx} & 0 \\ 0 & f_{yy} \end{pmatrix} \\
\mathbf{v}^\dagger\mathbf{A}\mathbf{v} &= \begin{pmatrix} \delta x & \delta y \end{pmatrix} \begin{pmatrix} f_{xx} & 0 \\ 0 & f_{yy} \end{pmatrix} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} \\
&= f_{xx}(\delta x)^2 + f_{yy}(\delta y)^2
\end{aligned}$$

In this case, it is clear that $\mathbf{v}^\dagger\mathbf{A}\mathbf{v}$ is always strictly positive for $\mathbf{v} \neq \mathbf{0}$ if and only if $f_{xx} > 0$ and $f_{yy} > 0$. Similarly, $\mathbf{A}$ is negative if and only if $f_{xx}, f_{yy} < 0$, and indefinite if one is negative and the other positive. (This is where we omit the case of a zero on the diagonal - we would have to consider higher-order behaviour in that direction in order to make any useful deductions.)

It is interesting that, in this case, only the behaviour of f along the two lines $f(x, y_0)$ and $f(x_0, y)$

is relevant - knowing the values of the second derivative along the axes allows us to deduce this value along any other axis, just like in the case of $\boldsymbol{\nabla} f$ and the first derivatives. It is natural to wonder whether there is in general a single pair of numbers (or a single pair of axes) associated with the matrix that determine whether it is positive definite and so on - in fact, there is a natural generalization of this to eigenvalues and eigenvectors.

Since $\mathbf{A}$ is real and symmetric (if $f_{xy} = f_{yx}$), it can be diagonalized (see the course on Vectors and Matrices - this is the *spectral theorem*) as it has a full set of orthogonal eigenvectors with associated eigenvalues λ_i. In the general case, writing vectors in the basis of eigenvectors, we always have

$$\begin{aligned} \delta\mathbf{x}\cdot\mathbf{H}\cdot\delta\mathbf{x} &= \begin{pmatrix}\delta x & \delta y \cdots \delta z\end{pmatrix}\begin{pmatrix}\lambda_x & 0 & \cdots & 0\\ 0 & \lambda_y & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \lambda_z\end{pmatrix}\begin{pmatrix}\delta x\\ \delta y\\ \vdots\\ \delta z\end{pmatrix}\\ &= \lambda_x(\delta x)^2 + \lambda_y(\delta y)^2 + \cdots + \lambda_z(\delta z)^2 \end{aligned}$$

so we have the following key result:

Lemma 2.8. $\mathbf{H}$ *is positive definite* $\iff$ *all eigenvalues* $\lambda_x, \cdots, \lambda_z$ *are positive. Similarly for the negative definite case.*

In order to avoid having to explicitly work out the eigenvalues of the matrix each time we do this, the following result, called *Sylvester's criterion*, is very useful:

Lemma 2.9 (Sylvester's criterion)**.** *An* $n \times n$ *real symmetric*[1] *matrix is positive definite* $\iff$ *the determinants of the leading minors, or the upper left* 1×1, 2×2, $\cdots$, *and* $n \times n$ *matrices, are all positive.*

So for example, in the case of a two dimensional Hessian with non-zero determinant (so it has no zero eigenvalues), to test for a minimum it is enough to test that $f_{xx} > 0$ and

$$\begin{vmatrix} f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{vmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} > 0$$

Similarly (as can be seen by just negating the entire matrix $\mathbf{H}$) the matrix is negative definite if and only if the first determinant is negative, $f_{xx} < 0$, the second is positive, and so on in an alternating fashion. Finally, any other pattern than $+ + + + \cdots$ and $- + - + \cdots$ results in an indefinite matrix. Combining these results, we have:

Theorem 2.10. *The pattern of signs of the determinants in a Hessian at a stationary point with non-zero determinant determines the nature of the point as follows:*

[1] Or more generally, Hermitian.

$+ + + + \cdots$	*The Hessian is positive definite, and the point is a minimum.*
$- + - + \cdots$	*The Hessian is negative definite, and the point is a maximum.*
other	*The Hessian is indefinite, and the point is a saddle point.*

Example 2.11. Find and categorize the stationary points of the function

$$f(x, y) = 8x^3 + 2x^2y^2 - x^4y^4 - 2 - 6x$$

First, we calculate the two first partial derivatives

$$\begin{aligned} f_x &= 24x^2 + 4xy^2 - 4x^3y^4 - 6 \\ f_y &= 4x^2y - 4x^4y^3 \end{aligned}$$

so that the gradient is

$$\boldsymbol{\nabla} f = \begin{pmatrix} 24x^2 + 4xy^2 - 4x^3y^4 - 6 \\ 4x^2y - 4x^4y^3 \end{pmatrix}$$

To find the stationary points, we need to solve $\boldsymbol{\nabla} f = \mathbf{0}$. From the $f_y = 0$ equation we have

$$\begin{aligned} 4x^2y - 4x^4y^3 &= 0 \\ 4x^2y\left(1 - x^2y^2\right) &= 0 \end{aligned}$$

and therefore $(x, y) = (0, ?), (?, 0), \left(x, \frac{1}{x}\right), \left(x, -\frac{1}{x}\right)$.

$x = 0$: $f_x = -6 = 0$. No solutions.

$y = 0$: $f_x = 24x^2 - 6 = 0$. Solutions $x = \pm\frac{1}{2}$.

$\left(x, \frac{1}{x}\right)$: $f_x = 24x^2 + 4\frac{1}{x} - 4\frac{1}{x} - 6 = 0$. Solutions $x = \pm\frac{1}{2}$ and $y = \pm 2$.

$\left(x, -\frac{1}{x}\right)$: $f_x = 24x^2 + 4\frac{1}{x} - 4\frac{1}{x} - 6 = 0$ again. Solutions $x = \pm\frac{1}{2}$ and $y = \mp 2$.

Hence the six stationary points are located at $\left(\pm\frac{1}{2}, 0\right)$, $\left(\pm\frac{1}{2}, \pm 2\right)$ and $\left(\pm\frac{1}{2}, \mp 2\right)$.

Now to classify these points, we must calculate the Hessian:

$$\begin{aligned} \mathbf{H} = \boldsymbol{\nabla}\boldsymbol{\nabla} f &= \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} \\ &= \begin{pmatrix} 48x + 4y^2 - 12x^2y^4 & 8xy - 16x^3y^3 \\ 8xy - 16x^3y^3 & 4x^2 - 12x^4y^2 \end{pmatrix} \\ &= 4\begin{pmatrix} 12x + y^2 - 3x^2y^4 & 2xy\left(1 - 2x^2y^2\right) \\ 2xy\left(1 - 2x^2y^2\right) & x^2\left(1 - 3x^2y^2\right) \end{pmatrix} \end{aligned}$$

Hence we have:

$\left(\frac{1}{2}, 0\right)$: Here

$$\mathbf{H} = 4\begin{pmatrix} 6 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

which is clearly positive definite, so this is a minimum.

$\left(-\frac{1}{2}, 0\right)$: Here

$$\mathbf{H} = 4\begin{pmatrix} -6 & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$$

which is clearly indefinite, so this is a saddle point.

$\left(\frac{1}{2}, 2\right)$: Here

$$\mathbf{H} = 4\begin{pmatrix} -2 & -2 \\ -2 & -\frac{1}{2} \end{pmatrix}$$

which has $f_{xx} < 0$ and a determinant which is also negative, and so is also indefinite, and hence this is a saddle point.

$\left(\frac{1}{2}, -2\right)$: Now

$$\mathbf{H} = 4\begin{pmatrix} -2 & 2 \\ 2 & -\frac{1}{2} \end{pmatrix}$$

which again is indefinite, so this is a third saddle point.

$\left(-\frac{1}{2}, 2\right)$: This time

$$\mathbf{H} = 4\begin{pmatrix} -14 & 2 \\ 2 & -\frac{1}{2} \end{pmatrix}$$

which has $f_{xx} < 0$ and a positive determinant - hence this is negative definite, and the point is a maximum.

$\left(-\frac{1}{2}, -2\right)$: Finally

$$\mathbf{H} = 4\begin{pmatrix} -14 & -2 \\ -2 & -\frac{1}{2} \end{pmatrix}$$

which is also negative definite, corresponding to a maximum.

Remark. We could have saved time by noting the symmetry of the function under $y \to -y$; the layout and character of stationary points must also be symmetrical across the x-axis.

In fact, this gives us all the information needed to make a sketch of the contours. Note that near a stationary point, the Taylor series tells us that the function is approximately of the form

$$f(\mathbf{x}) \approxeq f(\mathbf{x}_0) + \frac{1}{2}\left[(\delta x_1)^2 \lambda_1 + (\delta x_2)^2 \lambda_2\right]$$

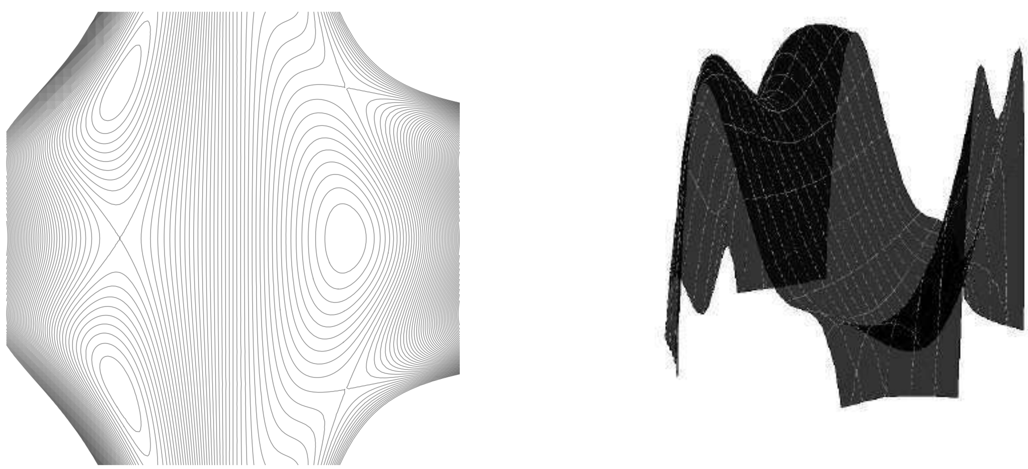
with respect to the principal axes (the eigenvectors) of the Hessian. So the contours are solutions of

$$\begin{aligned} f(\mathbf{x}) = f(\mathbf{x}_0) + \frac{1}{2}\left[(\delta x_1)^2 \lambda_1 + (\delta x_2)^2 \lambda_2\right] &= \text{constant} \\ (\delta x_1)^2 \lambda_1 + (\delta x_2)^2 \lambda_2 &= \text{constant} \end{aligned}$$

But clearly if λ_1, λ_2 both have the same sign, these are simply ellipses, rotated between the principal axes and the standard axes; and if they have different signs, then these are hyperbolae.

Having drawn the local behaviour of the contours near all stationary points, we must then fill in the space (and join up the hyperbolic lines) without creating any new stationary points - that is, no more contours must cross, and no more closed loops may be formed.

The actual contours and a three-dimensional plot of the function are shown in Figure 2.5.



(a) Contours (b) Surface

Figure 2.5: Classification of stationary points

2.3 Change of Variables

A very common task in mathematics is to transform the way we write down a problem via a *change of variables*. As with the chain rule, we already know how to deal with this in the single-variable case, because this is precisely the chain rule! We let $f(x) = f(x(t))$ and then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\mathrm{d}f}{\mathrm{d}x} \cdot \frac{\mathrm{d}x}{\mathrm{d}t}$$

so

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \tfrac{\mathrm{d}f}{\mathrm{d}t} / \tfrac{\mathrm{d}x}{\mathrm{d}t}$$

which is usually easily calculated as a function of t.

The multiple dimensional case, as might be expected, requires the multi-dimensional chain rule.

For example, if we wrote $x = x(r,\theta)$ and $y = y(r,\theta)$ as we would when converting from Cartesian coordinates (x, y) to polar coordinates (r, θ), then we would have f as a function of r and θ:

$$f = f(x(r,\theta), y(r,\theta))$$

Then the chain rule would give us

$$\left.\frac{\partial f}{\partial r}\right|_\theta = \left.\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right|_\theta$$

where we have explicitly stated, when taking the partial derivative of f with respect to r for example, that we are not holding x or y constant, but instead the accompanying variable θ.

Example 2.12. If $f = xy$ and $x = r\cos\theta$ and $y = r\sin\theta$, then

$$f = r^2 \sin\theta\cos\theta$$

and clearly

$$\left.\frac{\partial f}{\partial r}\right|_\theta = 2r\sin\theta\cos\theta$$

We can check the chain rule:

$$\begin{aligned}\left.\frac{\partial f}{\partial r}\right|_\theta &= \left.\frac{\partial f}{\partial x}\frac{\partial x}{\partial r}\right|_\theta + \left.\frac{\partial f}{\partial y}\frac{\partial y}{\partial r}\right|_\theta \\ &= y\cdot\cos\theta + x\cdot\sin\theta \\ &= r\sin\theta\cos\theta + r\sin\theta\cos\theta \\ &= 2r\sin\theta\cos\theta\end{aligned}$$

as expected.

2.4 Implicit Differentiation and Reciprocals

One of the other things we can now generalize is the idea of implicit differentiation. Classically, this means that we have an expression like

$$F(x, y) = \text{constant}$$

and we deduce that $\frac{\mathrm{d}F}{\mathrm{d}x} = 0$ for instance, using the chain rule to calculate

$$\frac{\mathrm{d}}{\mathrm{d}x}(xy) = y + x\frac{\mathrm{d}y}{\mathrm{d}x}$$

and so on.

Now imagine a surface in three-dimensional space, specified by

$$F(x, y, z) = \text{constant}$$

over space. We can write this as

$$F(x, y, z(x, y)) = \text{constant}$$

though this is a slight abuse of notation because $z(x, y)$ is *not* necessarily single-valued there may be multiple points above and/or below the point $(x, y, 0)$ in the xy-plane.

The differential form of the chain rule (again best used as a mnemonic) is

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z$$

Then at any such point, by the chain rule, we get

$$\left.\frac{\partial F}{\partial x}\right|_y = \left.\frac{\partial F}{\partial x}\frac{\partial x}{\partial x}\right|_y + \left.\frac{\partial F}{\partial y}\frac{\partial y}{\partial x}\right|_y + \left.\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right|_y$$

where the terms like $\partial F/\partial x$ have both y and z held constant. Clearly,

$$\left.\frac{\partial x}{\partial x}\right|_y = 1$$

and

$$\left.\frac{\partial y}{\partial x}\right|_y = 0$$

so this gives us

$$\left.\frac{\partial F}{\partial x}\right|_y = \frac{\partial F}{\partial x} + \left.\frac{\partial F}{\partial z}\frac{\partial z}{\partial x}\right|_y$$

and hence

$$\left.\frac{\partial z}{\partial x}\right|_y = -\frac{\partial F/\partial x}{\partial F/\partial z}$$

where both terms on the right have the variables not involved held constant (so y, z for the top term and x, y for the bottom term).

It is very important to note the introduction of the negative sign here: there is *not* a simple algebraic manipulation giving rise to this relationship ('cancelling the ∂F terms' for example).

These give rise to the interesting relationship that for any 2D surface in 3D space,

$$\boxed{\left.\frac{\partial x}{\partial y}\right|_z \left.\frac{\partial y}{\partial z}\right|_x \left.\frac{\partial z}{\partial x}\right|_y = -1}$$

Reciprocals

In the same sort of way that the above negative sign confounds our expectations from single-variable theory, the rules for inverting partial derivatives are not entirely obvious.

The normal reciprocal rules do hold *provided we keep the same variables constant.*

For example, in the transformation $(x, y) \to (r, \theta)$ we have

$$\frac{\partial r}{\partial x} \neq \frac{1}{\partial x / \partial r}$$

because on the left hand side we are assuming that y is held constant, whilst on the right hand side we are assuming that θ is held constant.

The correct statement would be

$$\left.\frac{\partial r}{\partial x}\right|_y = \frac{1}{\left.\partial x / \partial r\right|_y}$$

which is an altogether different statement. The meaning of the term on the bottom of the right hand side would be 'how fast does x change as I increase r at a steady rate, given that I also adjust θ so that y remains constant?'

Example 2.13. To see this explicitly for the case of polar coordinates, write $x = r\cos\theta$ and $y = r\sin\theta$ so that $r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}\left(y/x\right)$. Then

$$\begin{aligned} \left.\frac{\partial r}{\partial x}\right|_y &= \frac{x}{\sqrt{x^2+y^2}} \\ &= \frac{r\cos\theta}{r} \\ &= \cos\theta \end{aligned}$$

and

$$\left.\frac{\partial x}{\partial r}\right|_y = \left.\frac{\partial\left(r\cos\theta\right)}{\partial r}\right|_y$$

and if $y = r\sin\theta$ is constant, then $\sin\theta = \frac{y}{r}$ so $\cos\theta = \sqrt{1-\left(\frac{y}{r}\right)^2}$. Hence we can calculate

$$\begin{aligned} \left.\frac{\partial x}{\partial r}\right|_y &= \left.\frac{\partial\left(r\sqrt{1-\left(\frac{y}{r}\right)^2}\right)}{\partial r}\right|_y \\ &= \left.\frac{\partial\left(\sqrt{r^2-y^2}\right)}{\partial r}\right|_y \\ &= \frac{r}{\sqrt{r^2-y^2}} \\ &= \frac{1}{\sqrt{1-\left(\frac{y}{r}\right)^2}} \\ &= \frac{1}{\sqrt{1-\sin^2\theta}} \\ &= \frac{1}{\cos\theta} \end{aligned}$$

as required. By contrast,

$$\left.\frac{\partial x}{\partial r}\right|_{\theta} = \cos\theta \not\equiv \frac{1}{\cos\theta}$$

2.5 Differentiation of Integrals with Respect to Parameters

Consider as family of functions $f = f(x, c)$, for which we have a different graph $f = f_c(x)$ for each c.

Then we can define a corresponding family of integrals,

$$I(b, c) = \int_0^b f(x, c)\,\mathrm{d}x$$

Then by the fundamental theorem of calculus,

$$\frac{\partial I}{\partial b} = f(b, c)$$

To calculate the rate of change with respect to c we do the following:

$$\begin{aligned}
\frac{\partial I}{\partial c} &= \lim_{\delta c \to 0} \frac{1}{\delta c}\left[\int_0^b f(x, c + \delta c)\,\mathrm{d}x - \int_0^b f(x, c)\,\mathrm{d}x\right] \\
&= \lim_{\delta c \to 0}\left[\int_0^b \frac{f(x, c + \delta c) - f(x, c)}{\delta c}\mathrm{d}x\right] \\
&= \int_0^b \left.\frac{\partial f}{\partial c}\right|_x \mathrm{d}x
\end{aligned}$$

assuming that we are allowed to exchange limits and integrals like this (this result is actually always valid if both f and $\partial f/\partial c$ are continuous over the region of integration $[0, b]$, and the region of c in which we take the derivative[2]).

So if we take

$$I(b(x), c(x)) = \int_0^{b(x)} f(y, c(x))\,\mathrm{d}y$$

then we get, via the chain rule,

$$\begin{aligned}
\frac{\mathrm{d}I}{\mathrm{d}x} &= \frac{\partial I}{\partial b}\frac{\mathrm{d}b}{\mathrm{d}x} + \frac{\partial I}{\partial c}\frac{\mathrm{d}c}{\mathrm{d}x} \\
&= f(b, c)\,b'(x) + c'(x)\int_0^b \left.\frac{\partial f}{\partial c}\right|_y \mathrm{d}y
\end{aligned}$$

[2] This result is called the *Leibniz integral rule*, or *Leibniz's rule for differentiation under the integral sign*. A sophisticated result called the *Dominated convergence theorem* gives the general case for a more sophisticated type of integral called the *Lebesgue integral* (we are using the Riemann integral).

For example, if $I(x) = \int_0^x f(x, y)\, \mathrm{d}y$ then

$$\frac{\mathrm{d}I}{\mathrm{d}x} = f(x, x) + \int_0^x \left.\frac{\partial f}{\partial x}\right|_y \mathrm{d}y$$

---

3 First-Order Equations

In this section, we will consider both differential equations and difference equations (also known as recurrence relations) of the *first-order*, in which no more than one derivative, or one previous term of a sequence, appears.

It will be very useful to have a firm grasp of one particular function:

3.1 The Exponential Function

Consider $f(x) = a^x$, for some constant $a > 0$.

The rate of change of such a map can be calculated as follows:

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0}\frac{a^{x+h}-a^x}{h} &= \lim_{h\to 0}\frac{a^x\left(a^h-1\right)}{h} \\ &= a^x \lim_{h\to 0}\frac{a^h-1}{h} \\ &= a^x\cdot\lambda \end{aligned}$$

for some constant λ (independent of x) - note that this limit must converge to some value, since this map is obviously differentiable[3].

> **Definition 3.1.** The function $\exp x \equiv e^x$ is defined by choosing a so that $\lambda = 1$, i.e. $\frac{\mathrm{d}f}{\mathrm{d}x} = f$. We write $e = a$ for this case.

Remark. Let the inverse of the function $u = e^x$ be given by $x = \ln u$. Then if we write $y = a^x = e^{x\ln a}$ it becomes clear via the chain rule that

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= (\ln a)\, e^{x\ln a} \\ &= (\ln a)\, a^x \\ &= (\ln a)\, y \end{aligned}$$

so that $\lambda = \ln a$.

One other limit we will make use of is

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$$

Exercise 3.2.

(i) Prove that, for $x > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}\ln x = \frac{1}{x}$$

(*Hint*: Use the inverse function rule.)

[3] We do not show this formally in this course - in Analysis I we consider a^x to be defined in terms of e^x, and define e^x in terms of its *power series* $e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3!}x^3 + \cdots$, and derive all of these properties from here.

(ii) Write down the first few terms in the Taylor expansions of e^x and $\ln(1+x)$.

(iii) Use the Taylor expansion of the following expression to evaluate it:

$$\lim_{n\to\infty} \ln\left(1+\frac{x}{n}\right)^n$$

(iv) Why does it follow that

$$\lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n = e^x?$$

Using the fundamental theorem of calculus, and the first result from this exercise, we have that

$$\int_a^b \frac{1}{x}\,\mathrm{d}x = [\ln x]_a^b$$

if a and b are both strictly positive.

Now if a and b are both negative, then we can compute the integral either by symmetry, or formally by the change of variables $u = -x$,

$$\begin{aligned}
\int_a^b \frac{1}{x}\,\mathrm{d}x &= \int_{-a}^{-b} \frac{1}{(-u)} \cdot (-1) \cdot \mathrm{d}u \\
&= \int_{-a}^{-b} \frac{1}{u}\,\mathrm{d}u \\
&= \ln(-b) - \ln(-a) \\
&= \ln|b| - \ln|a| \\
&= [\ln|x|]_a^b
\end{aligned}$$

Because of these two facts, we commonly write

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x|$$

However, this assumes that x does not change sign or pass through 0 over the region of integration - if it does, then the integral is undefined.

A better way of writing this is

$$\int_a^b \frac{1}{x}\mathrm{d}x = \ln\left(\frac{b}{a}\right)$$

which is valid for the same a and b but which avoid using the modulus signs $|x|$. A very useful result which holds in general for valid complex paths (not passing through 0) is that

$$e^{\int_a^b \frac{1}{x}\mathrm{d}x} = \frac{b}{a}$$

Example 3.3. Both integrals in

$$\int_1^2 x^{-1}\mathrm{d}x = \int_{-1}^{-2} x^{-1}\mathrm{d}x = \ln 2$$

are defined, but

$$\int_{-1}^{2} x^{-1}\mathrm{d}x$$

is not.

3.2 First-Order Linear ODEs

It is often best to begin a new topic with an example, so that is exactly what we shall do.

Example 3.4. Solve $5y' - 3y = 0$.

We can easily solve this equation because it is separable:

$$\begin{aligned}
\frac{y'}{y} &= \frac{3}{5} \\
\int \frac{\mathrm{d}y}{y} &= \int \frac{3}{5}\mathrm{d}t \\
\ln|y| &= \frac{3}{5}t + C \\
y &= De^{\frac{3}{5}t}
\end{aligned}$$

This is the only function of this form. $y = Ae^{3/5x}$ is a solution for any real A, including $A = 0$ so $y = 0$. In this family of solution curves, or *trajectories* for one-dimensional cases like this, it is possible to pick out one *particular solution* using a boundary condition, like $y = y_0$ at $x = 0$, so that $A = y_0$.

In fact, *there are no other solutions.* To see this, let $u(t)$ be any solution, and compute the derivative of $ue^{-\frac{3}{5}t}$:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(ue^{-\frac{3}{5}t}\right) &= e^{-\frac{3}{5}t}\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{3}{5}e^{-\frac{3}{5}t}u \\
&= e^{-\frac{3}{5}t}\left[\frac{\mathrm{d}u}{\mathrm{d}t} - \frac{3}{5}u\right] \\
&= 0
\end{aligned}$$

It is a key result from our fundamental work that $ue^{-\frac{3}{5}t}$ is therefore a constant k, so indeed $u = ke^{\frac{3}{5}t}$ as required. It follows that there is a unique solution if we are given a boundary condition like those above.

It is not hard to guess, from the way that the numbers 3 and 5 appeared in our solution, that any similar equation has a solution of the form $e^{\lambda t}$, and in fact all solutions are of the form $ke^{\lambda t}$. In fact, any *linear, homogeneous ODE with constant coefficients* has families of solutions of the form $e^{\lambda t}$. We will define what all of these terms mean:

Definition 3.5. An nth order *linear* ODE has the form

$$c_n(x)\frac{\mathrm{d}^n y}{\mathrm{d}x^n}+c_{n-1}(x)\frac{\mathrm{d}^{n-1}y}{\mathrm{d}x^{n-1}}+\cdots+c_1(x)\frac{\mathrm{d}y}{\mathrm{d}x}+c_0(x)\,y=f(x)$$

A *homogeneous* equation has $f(x)\equiv 0$ so that $y=0$ is a solution.

A linear ODE with *constant coefficients* has $c_i(x)=c_i(0)=\text{constant}$ for all i.

There are a few important properties of such equations. The following two are the most important for us:

(i) Linearity and homogeneity mean that any multiple of a solution is another solution, as is the sum of any two solutions - that is, any *linear combination* of solutions is another solution. (Hence functions like $y=Ae^{\lambda_1 x}+Be^{\lambda_2 x}+\cdots$ will always be a solution if the corresponding basic terms are.)

(ii) An nth order linear differential equation has (only) n linearly independent solutions - that is, given $n+1$ solutions we can always rewrite one of them as a linear combination of the others. (Recall $y=Ae^{3/5x}$ was the *general solution* of the above first-order equation.)

The first fact is easy to prove, whereas the second is not obvious. We will begin to see how to talk about independence in section 4.3, and prove in section 6 all the results we need about solutions to higher-dimensional equations. For the case of first-order equations, we will briefly discuss existence and uniqueness of solutions in section 3.7. For now, however, we will leave these ideas to one side.

It is, however, useful to see why the solutions can be expressed in exponential form. The key idea is that of an *eigenfunction* of a differential operator (which is basically the left-hand side of the above). For our purposes:

Definition 3.6. A *differential operator* $D[y]$ acts on a function $y(x)$ to give back another function by differentiation, multiplication and addition of that function - for example

$$D[y]=x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-3y.$$

An *eigenfunction* of a differential operator D is a function y such that

$$D[y]=\lambda y$$

for some constant λ, which is called the *eigenvalue*.

Remark. The idea is very like that of eigenvectors and eigenvalues of matrices.

The important point to realize is that $y=e^{\lambda x}$ is an eigenfunction of our first-order linear differential operators with constant coefficients, because

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{\lambda x}=\lambda e^{\lambda x}$$

So in solving $ay' + by = 0$ all we need to do is solve

$$(a\lambda + b)\, e^{\lambda x} = 0$$

which gives

$$\lambda = -\frac{b}{a}$$

as we found above.

All we are really doing in solving these unforced equations is trying to find eigenfunctions with eigenvalue 0, to give the zero on the right-hand side[4].

3.2.1 Discrete equation

The above equation

$$5y' - 3y = 0$$

(say with the boundary condition $y = y_0$ at $x = 0$) has analogous discrete equations in the form of difference equations, where we solve for the values of some sequence y_n meant to approximate y at time steps like $y_n = y\,(nh)$.

The so-called *(simple) Euler approximation* substitutes $y \leftrightarrow y_n$ and $y' \leftrightarrow \frac{y_{n+1} - y_n}{h}$ where we take discrete steps of size h, giving $x = nh$.

This gives us

$$\begin{aligned} 5\frac{y_{n+1} - y_n}{h} - 3y_n &= 0 \\ y_{n+1} &= \left(1 + \frac{3h}{5}\right) y_n \\ y_n &= \left(1 + \frac{3h}{5}\right)^n y_0 \end{aligned}$$

Using our equation for the step size, we note $h = \frac{x}{n}$ so that we can eliminate h and return to having a dependence on

$$y\,(x) = y_n = y_0 \left(1 + \frac{3}{5}\frac{x}{n}\right)^n$$

and if we now take the limit $h \to 0$ or equivalently $n \to \infty$, so that we refine the step size, we retrieve

$$\begin{aligned} y\,(x) &= y_0 \lim_{n \to \infty} \left(1 + \frac{3x}{5} \cdot \frac{1}{n}\right)^n \\ &= y_0 e^{\frac{3x}{5}} \end{aligned}$$

which is the same as the equation we originally established (see 3.2). This is not very surprising, because the limit $h \to 0$ corresponds, in the equation we are solving, to the limiting equation $5\frac{\mathrm{d}y}{\mathrm{d}x} - 3y = 0$.

[4]If you know some linear algebra (like the material from the Vectors & Matrices course), then you might find it interesting to think of this as trying to find a *basis* for the *kernel* or *null-space* of the differential operator D.

3.2.2 Series solution

Another way of finding a solution (if a solution of this form exists - see the section on series solutions later in this course) is to assume

$$y = \sum_{n=0}^{\infty} a_n x^n$$

so that we also have

$$y' = \sum_{n=0}^{\infty} a_n n x^{n-1}$$

Now if we take our equation $5y' - 3y = 0$ we note that we can rewrite this as

$$5\left(xy'\right) - 3x\left(y\right) = 0$$

(the *equidimensional* form of the equation, in which the bracketed terms all have the same dimensions, with derivatives with respect to x balanced by multiplications by x - though now we have non-constant coefficients) and hence the equation becomes

$$\begin{aligned} \sum a_n \left[5nx \cdot x^{n-1} - 3x \cdot x^n\right] &= 0 \\ \sum a_n x^n \left[5n - 3x\right] &= 0 \end{aligned}$$

Now in this equation, since the left side is identically 0 for all x, we can compare the coefficients of x^n. This gives

$$5n \cdot a_n - 3a_{n-1} = 0$$

for all n including $n = 0$ (if we write $a_{-1} = 0$).

$n = 0$: The first equation we get from this is that $0 \cdot a_0 = 0$, which symbolizes the fact that we may consider a_0 to be *arbitrary* - remember we have a constant A or y_0 in our other solutions.

$n > 0$: In this case, we can divide through by n to obtain

$$\begin{aligned} a_n &= \frac{3}{5n} a_{n-1} \\ &= \frac{3}{5n} \cdot \frac{3}{5\left(n-1\right)} a_{n-2} \\ &= \cdots \\ &= \left(\frac{3}{5}\right)^n \cdot \frac{1}{n!} a_0 \end{aligned}$$

Hence we have

$$y = a_0 \sum_{n=0}^{\infty} \left(\frac{3}{5}\right)^n \cdot \frac{1}{n!} x^n$$

which is a valid expression for the solution. However, in this case, we have the good fortune to be able to identify it in closed form:

$$\begin{aligned} y &= a_0 \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{3x}{5} \right)^n \\ &= a_0 e^{\frac{3x}{5}} \end{aligned}$$

Remark. In general, there is no reason to expect a closed-form solution, so the previous expression would suffice as an answer.

3.3 Forced (Inhomogeneous) Equations

There are a few ways to classify different forcing terms $f(x)$. We will look at 3 different types of forcing for equations with constant coefficients in this section:

(i) *Constant* forcing: e.g. $f(x) = 10$

(ii) *Polynomial* forcing: e.g. $f(x) = 3x^2 - 4x + 2$

(iii) *Eigenfunction* forcing: e.g. $f(x) = e^x$

We will solve each case with an example to illustrate how to handle these problems.

Remark. We will see in section 4.4 (which treats the second-order case) and later in 6.5 (which is the general treatment) that there are ways of solving any problem with an inhomogeneity with some cleverly chosen integrals.

3.3.1 Constant forcing

> **Example 3.7.** $5y' - 3y = 10$
>
> Note that there is guaranteed to be a *steady* or *equilibrium* solution, in this case given by the *particular solution* (*PS*) $y_p = -10/3$, so that $y_p' = 0$.
>
> It turns out that a general solution can be written
>
> $$y = y_c + y_p$$
>
> where y_c is the *complementary function* (*CF*) that solves the corresponding unforced equation - so since we already know $y_p = Ae^{\frac{3x}{5}}$, we have a general solution of
>
> $$y = Ae^{\frac{3x}{5}} - \frac{10}{3}$$
>
> The boundary conditions can then be applied to this general solution (*not* the complementary function).

So the general technique is to find the equilibrium solution which perfectly balances the forced equation, and then add on the general solution. This approach is actually fairly general.

3.3.2 Polynomial forcing

Example 3.8. $5y' - 3y = 3x^2 - 4x + 2$

It is hopefully clear that there is no constant solution to this equation, since the left-hand side must vary to give the polynomial behaviour on the right-hand side. But the above approach is suggestive: could we find a quadratic to match the right-hand side?

Let us assume $y_p = ax^2 + bx + c$ is a solution to this equation. Then

$$5y' - 3y = (-3a)\,x^2 + (10a - 3b)\,x + (5b - 3c)$$

so comparing coefficients,

$$\begin{aligned} -3a &= 3 \\ 10a - 3b &= -4 \\ 5b - 3c &= 2 \end{aligned}$$

which can be easily solved to give

$$\begin{aligned} a &= -1 \\ b &= -2 \\ c &= -4 \end{aligned}$$

Thus

$$y_p = -\left(x^2 + 2x + 4\right)$$

is a particular solution, and the general solution is

$$y = Ae^{\frac{3x}{5}} - \left(x^2 + 2x + 4\right)$$

This approach is easily extended to any polynomial - we just come up with a trial solution (sometimes called an *ansatz*, basically just an educated guess) which is a polynomial of the same order as the forcing term, and solve to find the right coefficients.

3.3.3 Eigenfunction forcing

One other type of problem we commonly get involved a forcing term which is actually an eigenfunction of the differential operator. We shall investigate this via a practical example, taking the opportunity to demonstrate the process of converting a physical problem into a differential equation.

Example 3.9. In a radioactive rock, isotope A decays into isotope B at a rate proportional to the number a of remaining nuclei of A, and B decays into C at a rate proportional to the corresponding variable b. Determine $b\,(t)$.

We have two time-varying variables, a and b. We know exactly how a varies over time, since we can write its evolution via a simple homogeneous differential equation whose solution we know, by introducing a positive *decay constant* k_a controlling how fast the $A \to B$ reaction occurs:

$$\begin{aligned} \frac{\mathrm{d}a}{\mathrm{d}t} &= -k_a a \\ a &= a_0 e^{-k_a t} \end{aligned}$$

where we have written $a(0) = a_0$.

The equation for the evolution of b is more complicated, because it explicitly depends on the evolution of a - introducing a new decay constant k_b, we obviously have a $-k_b b$ term in b', but we also have b increasing at the same rate as a decreases. Hence:

$$\begin{aligned} \frac{\mathrm{d}b}{\mathrm{d}t} &= k_a a - k_b b \\ \frac{\mathrm{d}b}{\mathrm{d}t} + k_b &= k_a a_0 e^{-k_a t} \end{aligned}$$

Now we know that the forcing term is an eigenfunction of the differential operator on the left hand side, and so we can try to find a particular integral which is a multiple of this function, with a coefficient determined by the eigenvalue. That is, we guess

$$b_p = De^{-k_a t}$$

Then if b_p is a solution of the above equation, we can cancel the $e^{-k_a t}$ terms to be left with

$$\begin{aligned} -k_a D + k_b D &= k_a a_0 \\ D(k_b - k_a) &= k_a a_0 \end{aligned}$$

Now obviously we have a problem if $k_b - k_a = 0$, since then this equation has no solution unless k_a or a_0 is zero (both of which correspond to trivial cases of the problem).

Assuming $k_b \neq k_a$: we can determine b via the complementary function $b_c = Ee^{-k_b t}$, with

$$\begin{aligned} b(t) &= b_p + b_c \\ &= \frac{k_a}{k_b - k_a} a_0 e^{-k_a t} + Ee^{-k_b t} \end{aligned}$$

and if $b = 0$ at $t = 0$, we have

$$b(t) = \frac{k_a}{k_b - k_a} a_o \left(e^{-k_a t} - e^{-k_b t}\right)$$

Note that we can also determine from this the value of $^b/_a$ over time without knowing a_0:

$$\frac{b}{a} = \frac{k_a}{k_b - k_a}\left[1 - e^{(k_a - k_b)t}\right]$$

However, we can see there are some in which this sort of approach does not work in quite the expected form - what happens if we have to produce a term which the differential operator annihilates? By this, we mean, for example, trying to solve the above problem with $k_b = k_a$ - then the $b_p = De^{-k_a t}$ guess will lead to a $0 = f(t)$ equation, with the adjustable parameter D disappearing. A simpler example would be solving an equation where like $y' - y = e^x$ where we know that the forcing term e^x is an eigenfunction of $y' - y$ with eigenvalue 0.

3.3.4 Resonant forcing

When we come to see second-order differential equations, we will see that this sort of forcing leads to what is called *resonance* in oscillatory systems - a system which would normally behave like a sine wave, for example, which is forced at its own frequency can be made to have the size of the oscillations grow over time. Mathematically, however, the same approaches to solving this problem can be used for any order of equation.

The main method we will demonstrate involves *detuning* the original equation, so that it has an eigenvalue λ very slightly different from zero, and then letting $\lambda \to 0$.

Example 3.10 (Detuning)**.** Find the general solution of $y' - y = e^x$ by detuning the equation.

The first step here is to substitute the forcing term with $e^{(\mu+1)x}$ where we think of μ as being very small but non-zero. We then want to find the particular solution for this term equation, which we know how to do:

$$\begin{aligned} y &= De^{(\mu+1)x} \\ y' - y &= [D(\mu+1) - D]\, e^{(\mu+1)x} \\ D(\mu+1) - D &= 1 \\ D &= \frac{1}{\mu} \end{aligned}$$

Hence

$$\begin{aligned} y &= \frac{1}{\mu} e^{(\mu+1)x} \\ &= \frac{1}{\mu} e^{\mu x} \cdot e^x \end{aligned}$$

Now to take the limit as $\mu \to 0$ of this is not possible, since $\frac{1}{\mu} \to \infty$ and $e^{\mu x} \to 1$, so it has no limit. In terms of the Taylor series in μ,

$$\begin{aligned} \frac{1}{\mu} e^{\mu x} \cdot e^x &= \mu^{-1} \left(1 + \mu x + \frac{1}{2}\mu^2 x^2 + \cdots\right) \cdot e^x \\ &= \mu^{-1} e^x + x \cdot e^x + o(\mu) \end{aligned}$$

But note that when we picked out a particular solution, it was fairly arbitrary that we assumed it was of the form $De^{(\mu+1)x}$ - we can easily add arbitrary multiples of solutions to the complementary equation. In particular, it is clear that

$$\frac{1}{\mu}e^{\mu x}\cdot e^{x}-\frac{1}{\mu}e^{x}$$

is a solution, since e^x solves the above equation. Then we have that

$$y = xe^x + o(\mu)$$

is always a solution to the above equation, and as this suggests, as $\mu \to 0$ we have a solution xe^x:

$$\begin{aligned}(xe^x)' - xe^x &= (xe^x + e^x) - xe^x \\ &= e^x\end{aligned}$$

as required.

In fact, this is the embodiment of a general principle - say we have any first-order equation

$$ay' + by = f(x)$$

for a, b constant, where $f(x)$ is a multiple of y_c, a solution of the complementary equation. Then note that

$$\begin{aligned}a(x\cdot y_c)' + bx\cdot y_c &= ay_c + axy_c' + bx\cdot y_c \\ &= x\underbrace{(ay_c' + by_c)}_{0} + ay_c \\ &= ay_c\end{aligned}$$

Now if this is really a first-order equation, $a \neq 0$, so if we have an equation forced by the eigenfunction y_c, then we can find a general solution by adding on some multiple of $x \cdot y_c$.

Remark. As noted at the start of this section, section 6.5 on the general method of variation of parameters gives us a general proof of this.

Example 3.11. Find the general solution of $2y' + 6y = 3\left(e^{-3x} + e^{3x}\right)$.

Note that the complementary function solves $2y_c' + 6y_c = 0$. Hence $y_c = Ae^{-3x}$.

Now for a particular solution, we can guess a solution of the form $c \cdot xe^{-3x} + d \cdot e^{3x}$ because e^{-3x} solves the complementary equation:

$$\begin{aligned}2y' + 6y &= 2ce^{-3x} - 6cxe^{-3x} + 6de^{3x} \\ &\quad +6cxe^{-3x} + 6de^{3x} \\ &= 2ce^{-3x} + 12de^{3x}\end{aligned}$$

Thus comparing coefficients, we choose $c = \frac{3}{2}$ and $d = \frac{1}{4}$ to obtain

$$\begin{aligned} y &= Ae^{-3x} + \frac{3}{2}xe^{-3x} + \frac{1}{4}e^{3x} \\ &= \left(A + \frac{3}{2}x\right)e^{-3x} + \frac{1}{4}e^{3x} \end{aligned}$$

3.4 Non-Constant Coefficients and Integrating Factors

In this section, we will consider methods for handling first-order linear equations with *non-constant coefficients*. The general form of such an equation is

$$a(x)\, y' + b(x)\, y = c(x)$$

We put such an equation into *standard form* by eliminating the coefficient of y' - that is, by dividing through by $a(x)$:

$$\boxed{y' + p(x)\, y = f(x)}$$

There are a various techniques which can be applied to these problems. A frequently very useful approach is to reduce it to a problem with constant coefficients. To do this, we could attempt to define a new variable z so that we can write this equation in the form $z' = g(x)$, eliminating the mixed term $p(x)\, y$. This method is called using an *integrating factor*.

Consider a new variable $z(x) = \mu(x)\, y(x)$. Then

$$\begin{aligned} z' &= \mu y' + \mu' y \\ &= \mu\left(f(x) - p(x)\, y\right) + \mu' y \\ &= \mu f(x) + y\left(\mu' - \mu p(x)\right) \end{aligned}$$

which is in the required form if and only if

$$\frac{\mathrm{d}\mu}{\mathrm{d}x} = \mu p$$

But this is a separable equation: we can integrate it as follows:

$$\begin{aligned} p &= \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x} \\ \int p\,\mathrm{d}x &= \int \frac{1}{\mu}\frac{\mathrm{d}\mu}{\mathrm{d}x}\mathrm{d}x \\ &= \int \frac{1}{\mu}\mathrm{d}\mu \\ &= \ln|\mu| + C \end{aligned}$$

Then we have an explicit expression for a suitable μ (noting the arbitrary multiplicative constant arising from the additive one in the integral) in the form

$$\mu = Ae^{\int p\,\mathrm{d}x}$$

Now we have

$$\begin{aligned} z' &= \mu f \\ z = \mu y &= \int \mu f \mathrm{d}x \end{aligned}$$

so we have a solution

$$\boxed{y(x) = \frac{1}{\mu} \int \mu f \mathrm{d}x}$$

where we can take

$$\boxed{\mu(x) = e^{\int p \mathrm{d}x}}$$

Example 3.12. Solve the equation $xy' + (1-x)y = 1$.

The first stage in solving these equations is to put them in standard form, which gives

$$y' + \left(\frac{1}{x} - 1\right) y = \frac{1}{x}$$

Hence, using an integrating factor

$$\begin{aligned} \mu &= e^{\int \left(\frac{1}{x} - 1\right) \mathrm{d}x} \\ &= e^{\ln x - x + C} \\ &= Axe^{-x} \end{aligned}$$

where for completeness we have included the constant $A = e^C$. (It cancels at the next stage.)

Hence we have

$$\begin{aligned} Axe^{-x}y &= \int Axe^{-x} \cdot \frac{1}{x} \mathrm{d}x \\ xe^{-x}y &= \int e^{-x} \mathrm{d}x \\ &= D - e^{-x} \end{aligned}$$

with final solution

$$\begin{aligned} y &= \frac{1}{x}\left[De^x - 1\right] \\ &= \frac{D}{x} \cdot e^x - \frac{1}{x} \end{aligned}$$

Remark. Solutions where the leading coefficient $a(x)$ has a zero at some point - as in the above example, where $a(x) = x$ which is obviously zero at $x = 0$ - often exhibit *singular* behaviour at these points (something investigated again in 4.6). For example, the above equation has solutions whose value goes to $\pm\infty$ as $x \to 0$.

In fact, if we demanded a solution that was finite everywhere, then we would be forced to take $D = 1$, as only then do we get

$$y = \frac{e^x - 1}{x} \to 1 \quad \text{as} \quad x \to 0$$

3.5 Non-Linear Equations

An even more general class of first-order equations allows the coefficients of $\frac{\mathrm{d}y}{\mathrm{d}x}$ and y to depend upon y as well as x.

The general form of such a non-linear equation is

$$\boxed{Q(x,y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x,y) = 0}$$

where we have no apparent 'forcing terms' as $P(x,y)$ can absorb them all (note P is constrained to be a multiple of y, as we can just multiply by $\frac{1}{y}$!).

As we have already seen in deriving the method of integrating factors, one class of equation can be easily solved:

3.5.1 Separable equations

The equation is separable if it can be rewritten in the form

$$\boxed{q(y)\frac{\mathrm{d}y}{\mathrm{d}x} = p(x)}$$

or perhaps more memorably (in differential form)

$$q(y)\,\mathrm{d}y = p(x)\,\mathrm{d}x$$

If both expressions are integrable, then this can solved by integration of both sides as we saw above:

$$\begin{aligned} \int p(x)\,\mathrm{d}x &= \int q(y)\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x \\ &= \int q(y)\,\mathrm{d}y \end{aligned}$$

although we do not necessarily get an expression for y in terms of x (this depends on whether we can invert the function which appears upon integrating q).

Example 3.13. Solve

$$x\frac{\mathrm{d}y}{\mathrm{d}x} + 1 = y^2$$

Away from $x = 0$ we may write

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2 - 1}{x}$$

and away from $y = \pm 1$ we have

$$\frac{1}{y^2 - 1} \cdot \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

and then we need to perform two integrations: $\int \frac{1}{x}\mathrm{d}x$ and $\int \frac{1}{y^2-1}\mathrm{d}y$.

Note now that we must separately note the two constant solutions, $y = 1$ and $y = -1$, which both satisfy the original equation.

The second integral is usually calculated by the method of *partial fractions*: we write

$$\begin{aligned} \frac{1}{y^2 - 1} &= \frac{1}{(y+1)(y-1)} \\ &= \frac{A}{y+1} + \frac{B}{y-1} \end{aligned}$$

and it can be found that $A = -1/2$ and $B = 1/2$.

So for ranges where y does not pass through ± 1, we have

$$\begin{aligned} \int \frac{1}{y^2 - 1}\mathrm{d}y &= \frac{1}{2}\int \left[-\frac{1}{y+1} + \frac{1}{y-1} \right] \mathrm{d}y \\ &= \frac{1}{2}\left[-\ln|y+1| + \ln|y-1| \right] \\ &= \frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| \end{aligned}$$

and so we have

$$\frac{1}{2}\ln\left|\frac{y-1}{y+1}\right| = \ln|x| + C_1$$

Note that the term in the modulus symbol on the left can only change sign at $y = \pm 1$, and similarly at $x = 0$ on the right. So in any region where our analysis is valid, this equation is too.

Now in this case it is actually possible to rearrange this into an equation for y:

$$\begin{aligned} \ln\left|\frac{y-1}{y+1}\right| &= 2\ln|x| + C_2 \\ &= \ln x^2 + C_2 \end{aligned}$$

Then we have

$$\left|\frac{y-1}{y+1}\right| = |C_3|\, x^2$$

Now we can also drop the other modulus signs by allowing C_3 to have arbitrary sign:

$$\begin{aligned} \frac{y-1}{y+1} &= \pm |C_3|\, x^2 \\ &= C_3 x^2 \end{aligned}$$

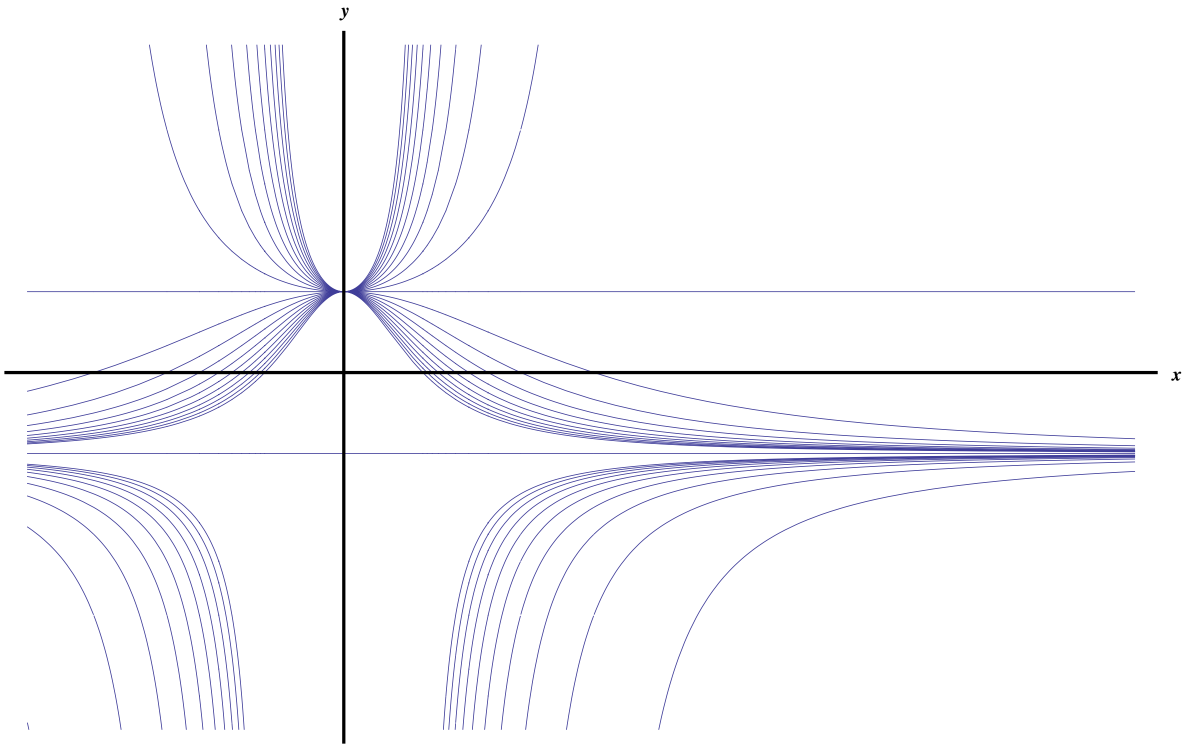


Figure 3.1: Solutions to Example 3.13 for $D = -1, -0.9, -0.8, \cdots, 0.9, 1$ plus the constant solution $y = -1$

which can be solved to get

$$y = \frac{1 + C_3 x^2}{1 - C_3 x^2}$$

So to summarize, there are two constant solutions, $y = \pm 1$. For y in $(-\infty, -1)$, or $(-1, 1)$, or $(1, \infty)$, there are solutions of the form

$$y = \frac{1 + Dx^2}{1 - Dx^2} = \frac{2}{1 - Dx^2} - 1$$

which may be joined continuously to any other part-solution at $y = \pm 1$ to form a correct global solution. (Similarly, at the singularity '$y = \pm\infty$' which occurs for $x = D^{-1/2}$ when $D > 0$, one can attach any other solution, and the result will also be a correct solution everywhere except at the singularity.) The two constant solutions along with a selection of other curves for positive and negative D are shown in Figure 3.1.

As in the previous section where we noted singular behaviour when the coefficient of y' was 0, we see here that at $x = 0$ the equation reduces to $1 = y^2$, and hence it is no longer a first-order equation. This behaviour results in an extra degree of freedom in the solution, corresponding to which solutions we 'glue' on at $x = 0$ (which corresponds to $y = 1$). (As mentioned before, some different types of singularity will be discussed in section 4.6.)

Note that because the original equation was invariant under the transformation $x \to -x$, for any solution $y(x)$, the reflection $y(-x)$ is also a solution. Similarly, because x only appeared in a so-called *equidimensional* term $x \cdot \mathrm{d}y/\mathrm{d}x$ which is invariant under $x \to ax$ for any $a \neq 0$, for any solution $y(x)$, the horizontal rescaling $y(ax)$ is also a solution if $a \neq 0$.

Note also the asymptotic behaviour as $x \to \pm\infty$ - apart from the *unstable* solution $y = 1$, which other solutions tend away from as $|x|$ grows, all solutions will converge in this limit to $y = -1$, a *stable* solution. We will consider this type of behaviour in section 3.6.

3.5.2 Exact equations

There is another wide class of first-order non-linear equations which can be solved reasonably straightforwardly. The equation

$$Q(x, y)\frac{\mathrm{d}y}{\mathrm{d}x} + P(x, y) = 0$$

is said to be an *exact* equation if and only if the differential form

$$\boxed{Q(x, y)\,\mathrm{d}y + P(x, y)\,\mathrm{d}x}$$

is an exact differential $\mathrm{d}f$ of some function $f(x, y)$.

If so, then the differential equation implies that $\mathrm{d}f = 0$, so $f = C$ is constant, which gives us an implicit relationship between x and y - that is, the solution.

Formally, we want to know that given any path $(x(t), y(t))$ in the xy-plane which satisfies the equation, then the function $f(x(t), y(t))$ is constant along that path. Hence, by the chain rule,

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t} = 0$$

so we want to identify

$$\begin{aligned} P &= \frac{\partial f}{\partial x} \\ Q &= \frac{\partial f}{\partial y} \end{aligned}$$

Now if this is true, then assuming that f is sufficiently nice that the order of second partial derivatives is irrelevant[5], we would have

$$\begin{aligned} \frac{\partial P}{\partial y} &= \frac{\partial^2 f}{\partial y \partial x} \\ \frac{\partial Q}{\partial x} &= \frac{\partial^2 f}{\partial x \partial y} \end{aligned}$$

and hence

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

[5] Recall Theorem 2.3 - if f has all second derivatives being continuous, then this holds.

making this a necessary condition for $P\mathrm{d}x + Q\mathrm{d}y$ to be an exact differential of such a function f.

We will state the converse without proof here:

Theorem 3.14. *If $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout a simply connected domain $\mathcal{D}$ then $P\mathrm{d}x + Q\mathrm{d}y$ is an exact differential of a single-valued function in $\mathcal{D}$.*

Remark. A *simply connected domain* is a space like the Euclidean plane, or a disc embedded in it, in which any two points have a path passing between them (it is *path-connected*) and such that any two different paths can be continuously deformed into each other. (So for example, a disc with a hole in the middle is not simply connected, because two paths passing on either side of the hole cannot be morphed into each other.)

If the equation is exact, then the solution $f =$ constant can be found by integrating both expressions

$$\begin{aligned} P &= \frac{\partial f}{\partial x} \\ Q &= \frac{\partial f}{\partial y} \end{aligned}$$

as we demonstrate here:

Example 3.15. Solve the equation

$$\cos(xy)\left[y + x\frac{\mathrm{d}y}{\mathrm{d}x}\right] + 2\left[x - y\frac{\mathrm{d}y}{\mathrm{d}x}\right] = 0$$

We can rewrite this as

$$(x\cos(xy) - 2y)\frac{\mathrm{d}y}{\mathrm{d}x} + (y\cos(xy) + 2x) = 0$$

and hence

$$\underbrace{(x\cos(xy) - 2y)}_{Q}\mathrm{d}y + \underbrace{(y\cos(xy) + 2x)}_{P}\mathrm{d}x = 0$$

Then we can see

$$\begin{aligned} \frac{\partial P}{\partial y} &= \cos(xy) - xy\sin(xy) \\ \frac{\partial Q}{\partial x} &= \cos(xy) - xy\sin(xy) \end{aligned}$$

so this is an exact equation.

Hence

$$\begin{aligned} \frac{\partial f}{\partial x} &= y\cos(xy) + 2x \\ f &= \sin(xy) + x^2 + C(y) \end{aligned}$$

where the 'constant' term C_1 is only constant with respect to x, but may vary with respect to y.

Now taking the derivative of this with respect to y we obtain

$$\frac{\partial f}{\partial y} = x\cos(xy) + C'(y)$$

but since we already know that $\frac{\partial f}{\partial y} = Q$ we have

$$\begin{aligned} x\cos(xy) + C'(y) &= x\cos(xy) - 2y \\ C'(y) &= -2y \\ C(y) &= -y^2 + D \end{aligned}$$

and hence

$$f = \sin(xy) + x^2 - y^2 + D$$

Thus the final solution is given by constant f; that is,

$$\sin(xy) + x^2 - y^2 = \text{constant}$$

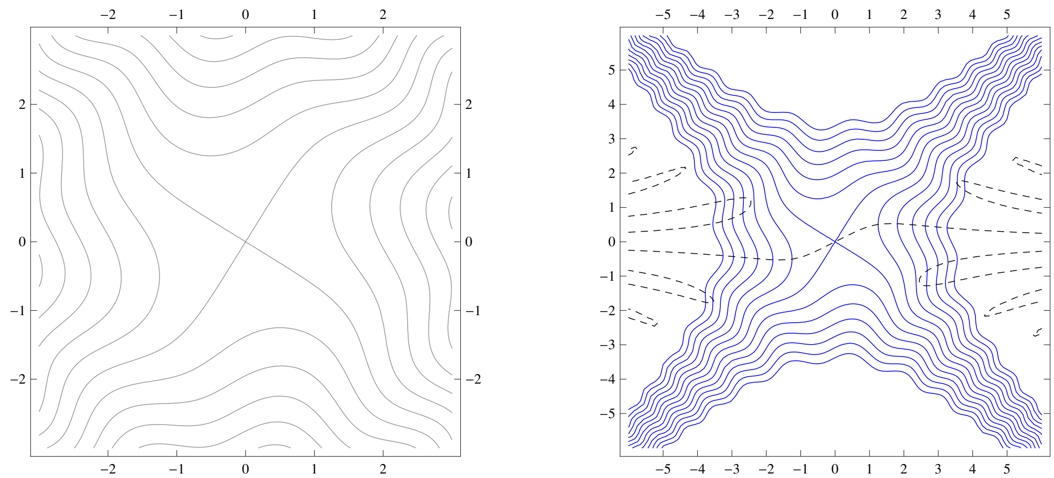
As we can see from the analytical solution, for large $|x|$ or $|y|$ the $\sin(xy)$ term becomes less dominant, and the solutions tend towards the hyperbolae $x^2 - y^2 = \text{const}$. Similarly, for small $|x|$ and $|y|$ the solution (working to first order in x and y independently) is roughly $xy + x^2 - y^2 = \text{constant}$ which are another set of hyperbolae at a different angle. All of this behaviour can be seen verified by visualizing the solutions.

A contour plot of f in the vicinity of the origin is displayed in Figure 3.2. Note that what we have shown is that the solution trajectories are constrained to move along these contours, since $\mathrm{d}f/\mathrm{d}t = 0$ along any acceptable path given by varying x and y. It does not necessarily follow, however, that any contour will give a *globally* valid solution to the original equation.

This is because we require it to be possible to parametrize the contour as $y(x)$, not just as $(x(t), y(t))$, so contours which 'double back' on themselves only satisfy the equation up until the point they reverse direction - at this point $\mathrm{d}y/\mathrm{d}x \to \pm\infty$ so we can expect to find singularities in the original differential equation. Indeed, the coefficient of $\mathrm{d}y/\mathrm{d}x$ is $(x\cos(xy) - 2y)$ which is 0 at precisely the points where the solutions fail.

Note also that the equation is singular at the origin, as it must be since two contours cross there! This is the same situation as we had previously, where either path leaving the origin is valid.

However, the equation is non-singular along the entire path if $(x\cos(xy) - 2y) \neq 0$ anywhere along it, a condition obeyed if $|2y| > |x|$, and in particular for any negative constant, as you may like to verify.



(a) Contour lines

(b) Singularities lie on the dotted lines

Figure 3.2: A few solutions to Example 3.15 near the origin, for various values of the constant.

3.6 Analysis of General First-Order Equations

In the previous section, we observed interesting behaviour of the analytical solutions to non-linear equations. It is natural to ask whether there are any general techniques for analyzing equations without solving them. Obviously one answer is that we use a computer to solve them numerically, using techniques like those demonstrated in section 3.2.1, where we turned the equation into a step-by-step process. However, this has two key problems:

- We can only numerically integrate one specific instance of a problem at a time - there may be complicated behaviours we don't observe because we don't try the correct parameters. The extreme case of this is in a *chaotic* system, where tiny changes in initial input result in massive changes in the behaviour over fairly short periods.
- It may not be possible to numerically integrate the equation accurately enough to examine its behaviour except over very short periods as the system has singular behaviour at some points.

This means that it is very useful to develop tools for analyzing an analytical problem in as much detail as possible before resorting to numerical approaches.

We will consider the general case of an equation of the form

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f\left(y, t\right)$$

Let us begin with a simple example of an equation that we can solve analytically.

3.6.1 Worked example of solution sketching

Example 3.16. Analyze the behaviour of solutions in case

$$f = t\left(1-y^2\right)$$

or $\dot{y} = t\left(1-y^2\right)$.

This equation is separable:

$$\begin{aligned} \int \frac{\mathrm{d}y}{1-y^2} &= \int t\mathrm{d}t \\ \frac{1}{2}\ln\left|\frac{1+y}{1-y}\right| &= \frac{t^2}{2} + C \\ \frac{1+y}{1-y} &= Ae^{t^2} \\ y &= \frac{Ae^{t^2}-1}{Ae^{t^2}+1} \end{aligned}$$

This gives us a parametrized family of solutions depending on the variable A. We can rewrite this as

$$y = 1 - \frac{2}{1+Ae^{t^2}}$$

and hence sketch it for say $A = 1$ - we consider $t > 0$ (treating this as a problem of evolution in time from initial conditions - we call the solution curve a *trajectory* when we look at it this way):

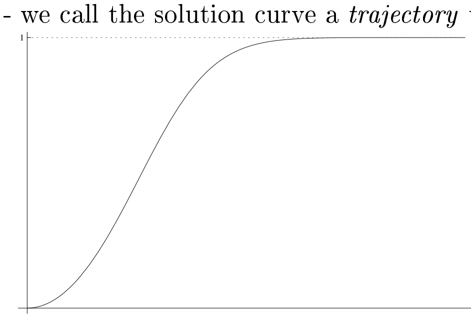


Figure 3.3: Solution curve for Example 3.16 for the case $A = 1$

So can we understand this equation without solving it?

(i) A natural first step is to find where $\dot{y} = f = 0$. In this case, this occurs at $t = 0$ and $y = \pm 1$.

From this it follows that $y = 1$ and $y = -1$ are both solutions.

Also, if we focus on $t > 0$, then we can actually note further that $\dot{y} < 0$ for $y > 1$ and $y < -1$, and $\dot{y} > 0$ for $y \in (-1, 1)$.

(ii) Another very useful idea for sketching solutions and seeing how they behave is to consider how the gradient field varies in space. One way to do this is to consider the *isoclines*, which are the lines along which f is constant (i.e. the contours of f).

In this case, we have

$$t\left(1-y^2\right) = C$$

and hence either $t = 0$, giving the y-axis, or

$$y^2 = 1 - \frac{C}{t}$$

We can separate this into the two cases $C > 0$ and $C < 0$, or increasing and decreasing y - note that $C = 0$ gives the isoclines $y = \pm 1$ along which the gradient is 0.

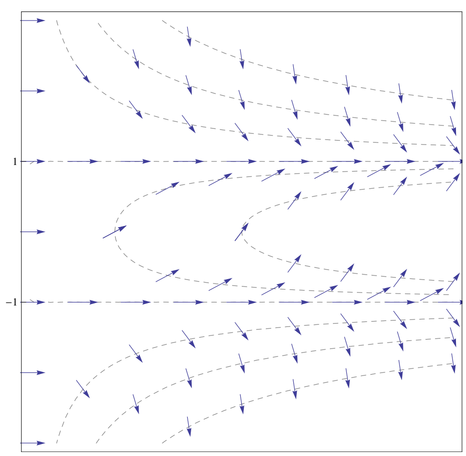
(a) If $C > 0$, then $y^2 \to 1$ from below as t grows, starting where $y^2 = 1 - C/t = 0$; i.e. at $t = C$. These lines form long 'c' shapes in between the two lines $y = \pm 1$.

(b) If $C < 0$, then y^2 descends from arbitrarily high near $t = 0$ to 1 as t grows.

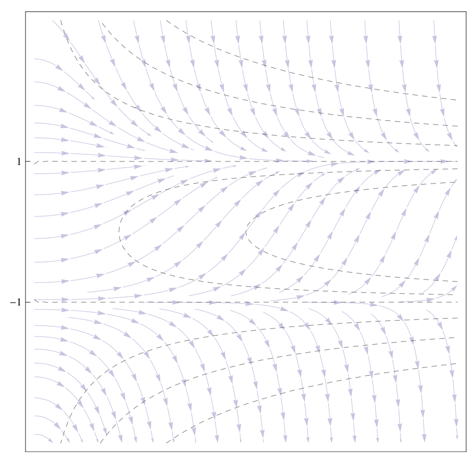
These lines are all sketched with arrows indicating the direction of the gradient (given by C) at that point to give a diagram like those shown in Figure 3.4a.

(iii) From the above, we can deduce that $y = 1$ is a *stable* solution, and $y = -1$ is an *unstable* solution, by seeing that the arrows around these two lines point towards and away from them respectively.

(iv) To sketch solution curves, we can simply join up the arrows we have drawn, and a selection of solutions is shown in Figure 3.4b.



(a) Isoclines with marked gradients



(b) Solution curves and their intersections with the isoclines

Figure 3.4: Isoclines and solutions (flow lines) for Example 3.16.

Note that if the function $f(y, t)$ giving the gradient is single-valued, solution curves *cannot cross*. This is because a single point on the curve is therefore sufficient to calculate $\mathrm{d}y/\mathrm{d}t$ at that point, and the whole solution can then be deduced by integrating away from this point.

3.6.2 Stability of equilibrium points

A natural question to ask about a solution, in light of the points made in introducing this section, is whether or not a small change in the value of the variables at some point decays over time, becoming insignificant, or grows to make the two solutions diverge significantly.

This is the idea of *stability* we have already seen in previous sections.

We will concern ourselves with fixed or equilibrium points here (although it is perfectly valid in general to talk about the stability of a more complicated solution), which are points where

$$\frac{\mathrm{d}y}{\mathrm{d}t} = f(y,t) = 0 \quad \text{for all } t$$

as in $y = \pm 1$ in our example above. Recall that $y = 1$ was stable, and $y = -1$ was unstable.

Perturbation analysis Imagine that we have located some fixed point $y = a$, so that $f(a,t) \equiv 0$.

Then we are concerned with the behaviour of the deviation from a over time. Therefore, we will consider an initially nearby solution $y(t) = a + \epsilon(t)$, where we assume $\epsilon(t)$ is small. Then

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}t} &= f(a+\epsilon(t), t) \\ &= f(a,t) + \epsilon \frac{\partial f}{\partial y}(a,t) + O\left(\epsilon^2\right) \\ &= \epsilon \frac{\partial f}{\partial y}(a,t) + O\left(\epsilon^2\right)\end{aligned}$$

This actually gives us an approximate differential equation for ϵ, so long as $\frac{\partial f}{\partial y}(a,t)$ is non-zero (see remark below), because clearly $\mathrm{d}y/\mathrm{d}t = \mathrm{d}\epsilon/\mathrm{d}t$ as a is a constant:

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}t} \approxeq \frac{\partial f}{\partial y}(a,t) \cdot \epsilon$$

Note that this is equation is *linear*, because the coefficient of ϵ is only a function of t (because we evaluate the partial derivative at $y = a$).

Example 3.17. Returning to the case $f = t\left(1 - y^2\right)$ we have

$$\begin{aligned}\frac{\partial f}{\partial y} &= -2yt \\ &= \begin{cases} -2t & \text{at } y = 1 \\ 2t & \text{at } y = -1 \end{cases}\end{aligned}$$

and hence:

- near $y = 1$,

$$\begin{aligned}\dot{\epsilon} &= -2t\epsilon \\ \epsilon &= \epsilon_0 e^{-t^2} \to 0 \quad \text{for any } \epsilon_0 \text{ as } t \to \infty\end{aligned}$$

and hence a sufficiently small perturbation ϵ (small enough that the linearization approximation is valid) will always decay to 0. We conclude that the fixed point $y = 1$ is stable.

- near $y = -1$,

$$\begin{aligned} \dot{\epsilon} &= 2t\epsilon \\ \epsilon &= \epsilon_0 e^{t^2} \to \pm\infty \quad \text{for any } \epsilon_0 \neq 0 \text{ as } t \to \infty \end{aligned}$$

so the perturbation ϵ grows as time passes, and the fixed point $y = -1$ is unstable. Note that when we say '$\epsilon \to \pm\infty$' we only mean that initially ϵ grows; in fact, once it is sufficiently large higher-order terms may dominate the expression for $\dot{\epsilon}$, as happens when a small positive displacement is made from $y = -1$. (We know that then $y \to 1$, the stable solution.)

Remark. As mentioned above, in the case that $\frac{\partial f}{\partial y}(a,t) = 0$, the approximation for $\dot{\epsilon}$ is not valid, and we need to take higher order terms in the Taylor expansion for f. Consider, for example, the unpleasant-looking equation $\dot{y} = \cos y - 1$. This can in fact be solved analytically to give $y = 2\cot^{-1}(t + C)$; in terms of $y(0) = y_0$, $C = \cot(y_0/2)$.

In this case, $\partial f/\partial y = -\sin y$ is identically 0 at the obvious equilibrium solution $y = 0$. Hence taking an extra term in the Taylor expansion, we have the 'approximation'

$$\begin{aligned} \dot{\epsilon} &\approxeq \frac{1}{2}\frac{\partial^2 f}{\partial y^2}(0,t) \cdot \epsilon^2 \\ &= \frac{1}{2} \cdot (-\cos 0) \cdot \epsilon^2 \\ &= -\frac{1}{2}\epsilon^2 \end{aligned}$$

which can be exactly solved for ϵ, though this is not in fact necessary: simply note that for any $\epsilon > 0$, $\dot{\epsilon} < 0$, and it is clear therefore that $\epsilon \to 0$, since $\dot{\epsilon} = 0$ at $\epsilon = 0$. Hence positive perturbations decay over time. But by contrast, if $\epsilon < 0$ then we also have $\dot{\epsilon} < 0$, so negative perturbations grow over time. As a result, the point is what is sometimes termed *semi-stable*.

This demonstrates a typical application of the theory. A particular special case arises for *autonomous systems*.

Definition 3.18. An *autonomous system* is one in which $\dot{y} = f(y)$ is independent of t.

Autonomous systems In this case, near a fixed point $y = a$,

$$\begin{aligned} y(t) &= a + \epsilon(t) \\ \dot{\epsilon} &\approxeq \left[\frac{\mathrm{d}f}{\mathrm{d}y}(a)\right]\epsilon \\ &= ke \end{aligned}$$

for some constant k as long as $k \neq 0$ (see the remark above). We know this has solution

$$\epsilon = \epsilon_0 e^{kt}$$

and hence the fixed point $y = a$ is stable or unstable according to whether $k = \frac{\mathrm{d}y}{\mathrm{d}t}(a)$ is negative or positive.

Example 3.19. One physical instance of stability problems like this occurs in considering chemical reactions.

For instance, consider the neutralization reaction in which sodium hydroxide (NaOH) and hydrochloric acid (HCl) in water react to form a new water molecule and the salt sodium chloride (NaCl).

	$NaOH$	+	HCl	$\xrightarrow{H_2O}$	H_2O	+	$NaCl$
number of molecules	a		b		c		c
initial number of molecules	a_0		b_0		0		0

If the reactants are in dilute solution in water, then the rate of reaction is proportional to the product of the numbers of reactants, ab. Hence

$$\begin{aligned} \frac{\mathrm{d}c}{\mathrm{d}t} &= \lambda ab \\ &= \lambda (a_0 - c)(b_0 - c) \\ &= f(c) \end{aligned}$$

Because this system is autonomous, as $f = f(c)$, we can simplify the plot greatly. In fact, we need only consider the plot of $f = \mathrm{d}c/\mathrm{d}t$ against c. Without loss of generality, assume $a_0 < b_0$:

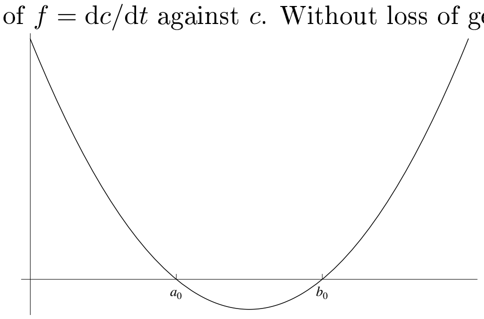


Figure 3.5: Plot of c against $f(c)$ for Example 3.19, in the case $a_0 < b_0$

Noting that points where the curve is above the c-axis correspond to increasing $c(t)$, and similarly points below to decreasing $c(t)$, we can form the so-called *phase portrait*:

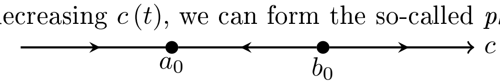


Figure 3.6: Phase portrait for $f(c)$ in Example 3.19

The arrows either side of a_0 point towards it, and the arrows either side of b_0 point away from it. Hence a_0 is stable, and b_0 is unstable.

In fact, in this chemical problem, it is clear that it is not a physical solution to have negative a or b and hence only the left-most portion of the diagram is relevant, showing that the system tends (beginning at any point corresponding to a possible stage in the reaction) towards the stable equilibrium at $c = a_0$ - that is, the reaction gradually slows to zero as all of chemical a is used up, as we would expect.

Remark. The phase portrait for a semi-stable point x_0 would look something like the following:

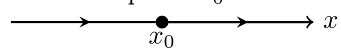


Figure 3.7: Example of a phase portrait for a semi-stable point

Exercise 3.20. Solve the above equation explicitly for $c(t)$.

3.6.3 The logistic equation

The final example of a first-order equation that we will consider is extremely well-known because of its interesting behaviour. It was originally presented as a simple model of population dynamics.

First, imagine a population of size y, where we assume y is large enough that taking it to be varying *continuously* in time is a suitable approximation. Suppose it is controlled by two parameters,

- a constant birth rate, so that y increases at a rate αy; and
- a constant death rate, so that y decreases at a rate βy.

Then obviously we have

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \alpha y - \beta y = (\alpha - \beta)\, y \\ y &= y_0 e^{(\alpha-\beta)t} \end{aligned}$$

so either y grows or decreases exponentially (or remains constant) depending only on whether $\alpha > \beta$ or $\alpha < \beta$ (or $\alpha = \beta$).

The problem here is that the death rate, for most realistic populations, is clearly going to be affected by the size of the population, due to effects like competition for limited resources.

So now imagine a population controlled by a birth rate and natural death rate proportional to y as before, but with the additional effect that individuals are competing for food. The probability that some source of food is found is proportional to y, and the probability of the *same* source of food being found by two individuals is proportional to y^2. If we assume that two individuals finding the same source of food fight to the death over it, then an extra term in the death rate appears, proportional to y^2:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = (\alpha - \beta)\, y - \gamma y^2$$

We conventionally rewrite this in terms of two new variables $r = \alpha - \beta$ and $K = \frac{r}{\gamma}$ as follows:

$$\boxed{\dot{y} = ry\left(1 - \frac{y}{K}\right)}$$

Here, r defines the growth rate of the population (the rate at which the population would grow in the absence of competition, assumed to be positive here, so that the species does not simply go extinct even in perfect conditions), and K is the *carrying capacity*: the maximum number of individuals that can be sustained indefinitely by the environment. Note that $\dot{y}$ changes sign at $y = K$, and indeed that $y = K$ is an equilibrium point.

This is called the *(differential) logistic equation*.

We can rewrite the equation in terms of the variable $x = y/K$, the ratio of the population to the carrying capacity, eliminating this parameter entirely:

$$\begin{aligned} \dot{x} &= \frac{1}{K}\dot{y} \\ &= \frac{ry}{K}\left(1 - \frac{y}{K}\right) \\ &= rx\,(1-x) \end{aligned}$$

We will analyze the equation in the scale-invariant form

$$\boxed{\dot{x} = rx\,(1-x)}$$

writing

$$\dot{x} = f\,(x)$$

In fact, we can easily form the phase portrait for these equations:

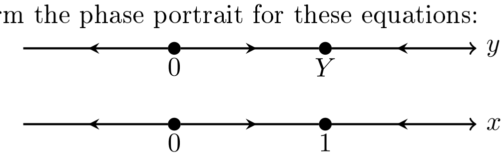


Figure 3.8: Phase portraits for the logistic equation

The logistic map It might be fruitful to think about changes of population of happening over discrete time; crudely speaking, for example, thinking of the births happening in spring, and the deaths in the winter.

So let us attempt to consider the analogous discrete-time version of the logistic equation by writing $x \to z_n$ and $\dot{x} \to (z_{n+1} - z_n)/\Delta t$, where Δt time passes between z_n and z_{n+1}.

This approximation of the time-derivative gives us the equation

$$\begin{aligned} \frac{z_{n+1} - z_n}{\Delta t} &= rz_n\,(1 - z_n) \\ z_{n+1} &= z_n + \Delta t \cdot rz_n\,(1 - z_n) \\ &= (1 + r\Delta t)\,z_n - r\Delta t \cdot z_n^2 \\ &= (1 + r\Delta t)\,z_n\left[1 - \left(\frac{r\Delta t}{1 + r\Delta t}\right) z_n\right] \end{aligned}$$

Now write $\lambda = 1 + r\Delta t$ and let

$$x_n = \left(\frac{r\Delta t}{1 + r\Delta t}\right) z_n$$

so we have

$$\boxed{x_{n+1} = \lambda x_n \left(1 - x_n\right)}$$

In this equation, x_n acts like the population, and λ like a time-adjusted version of r. We call this equation the *logistic map*.

We similarly write

$$x_{n+1} = f\left(x_n\right)$$

where $f(x) = \lambda x (1 - x)$ is in essentially the same form as for the continuous equation (we have merely transformed $r \to \lambda$) - note that it is not obvious that this relationship should hold: in the form $\dot{y} = f(y)$, $f(y)$ expresses a rate of change, whereas here, it gives the next value in the sequence.

Remark. We are taking $\lambda > 0$ here as well.

Behaviour of the logistic map It is possible to analyze the behaviour over time of a discrete first-order map like this by making so-called *cobweb diagrams*.

We begin by drawing a graph of x_n against x_{n+1}, in this case the parabola f, and then we can trace the route (or *orbit*) a point x_0 takes over time as follows:

(i) Find the point corresponding to x_0 on the x_n-axis.

(ii) Draw a line vertically until it intersects with the curve f.

(iii) Draw a line horizontally until it intersects with the line $x_{n+1} = x_n$.

(iv) Repeat steps 2 and 3 from the new point.

The idea is that the vertical line gives the point needed on the x_{n+1} axis, and we can then find the corresponding point on the horizontal x_n axis by using the line $x_{n+1} = x_n$.

For $\lambda < 1$, we have a graph like that shown in Figure 3.9a, for $x_0 > 0.5$.

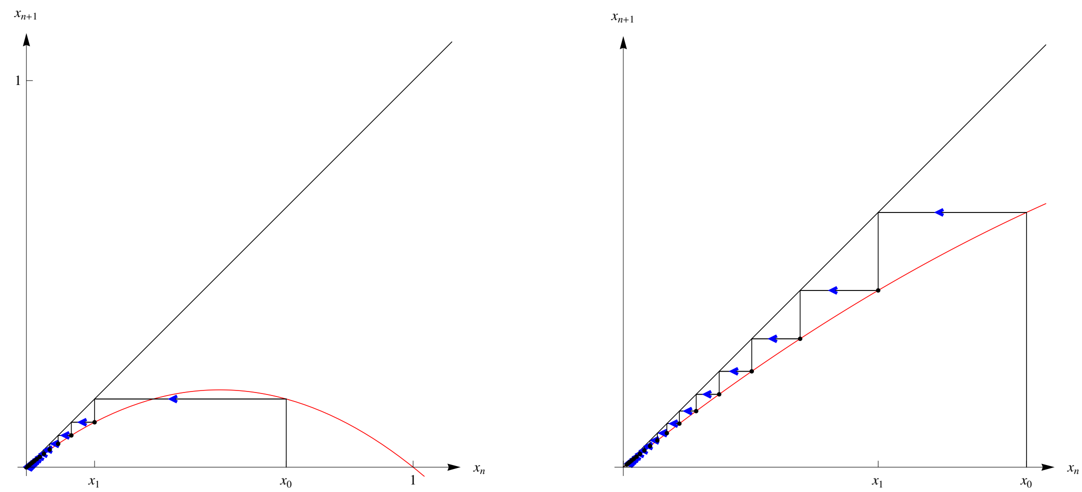
The diagram indicates that $x = 0$ is a stable fixed point. We can take this opportunity to find all fixed points for arbitrary λ. We require $x_{n+1} = f(x_n) = x_n$, so we want to solve

$$\begin{aligned} \lambda x (1 - x) &= x \\ x (\lambda (1 - x) - 1) &= 0 \\ x ((\lambda - 1) - \lambda x) &= 0 \end{aligned}$$

which has solutions

$$\begin{aligned} x &= 0, \frac{\lambda - 1}{\lambda} \\ &= 0, 1 - \frac{1}{\lambda} \end{aligned}$$

Note that for $\lambda < 1$ the second equilibrium point lies out side the interval $[0, 1]$ under consideration, but for $\lambda > 1$ it lies inside it. It appears in Figure 3.10.



(a) $0 < \lambda < 1$: Full diagram

(b) $0 < \lambda < 1$: Closer version with smaller x_0

Figure 3.9: Cobweb diagrams for the logistic map with $\lambda < 1$

Here, it appears that 0 is an unstable fixed point, whilst the new fixed point (at the intersection of $f(x_n) = x_n$) appears stable.

Before we continue with the cases of $\lambda > 2$, let us briefly consider the stability of fixed points in the general case of any first-order recurrence relation.

Stability of fixed points Suppose $x_n = X$ is a fixed point of the map $x_{n+1} = f(x_n)$. As before, consider a small perturbation ϵ, so $x_n = X + \epsilon_n$.

Then using a Taylor expansion of f we find that

$$\begin{aligned} X + \epsilon_{n+1} &= f(X + \epsilon_n) \\ &= f(X) + \epsilon_n f'(X) + O\left(\epsilon_n^2\right) \\ \epsilon_{n+1} &\approxeq \epsilon_n f'(X) \end{aligned}$$

since $X = f(X)$ by hypothesis, again with the assumption that $f'(X) \neq 0$.

Now X is stable if a small deviation gets smaller over time; that is, the magnitude of adjacent errors falls.

$$\left|\frac{\epsilon_{n+1}}{\epsilon_n}\right| < 1$$

which is equivalent (given a non-constant first-order approximation to f) to

$$|f'(X)| < 1$$

Similarly, if $|\epsilon_{n+1}/\epsilon_n| = |f'(X)| > 1$ then the point is unstable.

Note that this is valid for any first-order relation so long as f can be approximated by a Taylor expansion of first-order, and $f'(X) \neq 0$.

We can also deduce information from the sign of $\epsilon_{n+1}/\epsilon_n$. A positive ratio means that we expect

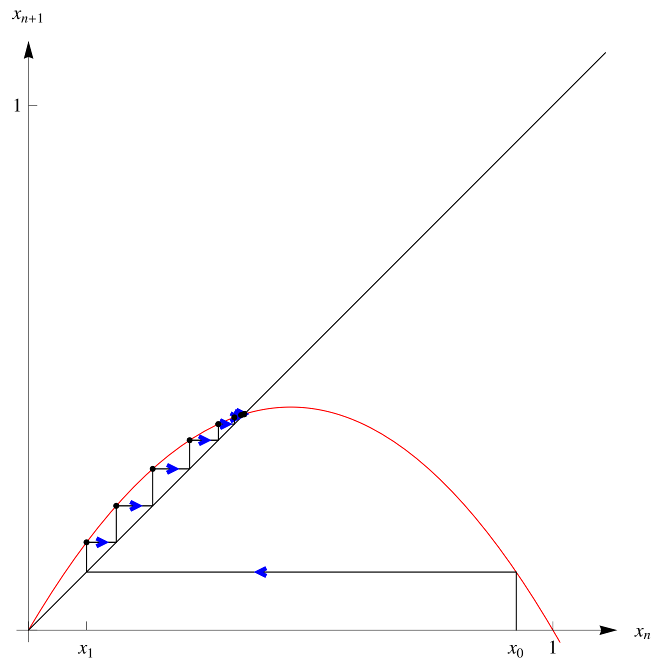


Figure 3.10: Cobweb diagram for the logistic map with $1 < \lambda < 2$

the sequence x_n to tend directly to any stable fixed point, since consecutive displacements have the same sign. Similarly, we expect oscillatory behaviour near any stable fixed point for the negative ratio, since a negative perturbation becomes positive and vice versa. (Indeed, where the fixed points are unstable, we can also expect local behaviour of the same kind.)

Remark. The $f'(X) = 0$ case can be dealt with as before, by considering the ratios obtained from higher-order terms in the Taylor series. If $|f'(X)| = 1$ then the problem is more complicated. To first-order in ϵ_n, the ratio is given by

$$\frac{\epsilon_{n+1}}{\epsilon_n} \approxeq f'(X) + \frac{1}{2}\epsilon_n f''(X)$$

so there are different cases according to the signs of f' and f'' at this point. (We consider only the $f''(X) \neq 0$ cases here.) If both are positive, then for positive displacements the ratio is slightly greater than 1, and for negative displacements it is between 0 and 1 for sufficiently small ϵ_n, so the point is semi-stable[6]. Similarly, if $f'(X) > 0$ and $f''(X) < 0$ then the point is semi-stable.

If $f'(X) < 0$ then $\epsilon_{n+1} = -\epsilon_n + \mu\epsilon_n^2$ for μ sharing the sign of $f''(X)$ - we assume $\mu > 0$ without loss of generality. The sequence ϵ_n obviously alternates in sign (for small ϵ_n) and hence if $\epsilon_n > 0$ then $|\epsilon_n| - |\epsilon_{n+1}| = \mu\epsilon_n^2$. Then $\epsilon_{n+2} > 0$ and $|\epsilon_{n+2}| - |\epsilon_{n+1}| = \mu\epsilon_{n+1}^2$. So $|\epsilon_{n+2}| - |\epsilon_n| = \mu\left(\epsilon_{n+1}^2 - \epsilon_n^2\right) < 0$ as $|\epsilon_{n+1}| < |\epsilon_n|$. We have $|\epsilon_{n+1}| < |\epsilon_{n+2}| < |\epsilon_n|$. It is clear that if $\epsilon_j > 0$ then $|\epsilon_{2m+j}|$ is a strictly decreasing sequence bounded below by zero, so $|\epsilon_{2m+j}| \to l$ for some l; in fact, because ϵ_{2m+j} shares

[6] We have $\epsilon_{n+1}/\epsilon_n = 1 + \mu\epsilon_n$ so for negative displacements, $\epsilon_n < \epsilon_{n+1} < 0$ and hence ϵ_n is a strictly increasing sequence bounded above, and therefore converges (a property of the real numbers; see Numbers & Sets or Analysis I) to some value l. Then taking limits in $\epsilon_{n+1} = \epsilon_n + \mu\epsilon_n^2$ we have $l = l + \mu l^2$ so $\mu l^2 = 0$ and hence $\epsilon_n \to l = 0$ as we assumed $\mu \neq 0$.

the same sign as ϵ_j, $\epsilon_{2m+j} \to l > 0$. Then evidently

$$\epsilon_{2m+j+1} = -\epsilon_{2m+j} + \mu\epsilon_{2m+j}^2 \to -l + \mu l^2$$

But we know that the limit of $\epsilon_{2(m+1)+j}$ is l, so

$$\begin{aligned} \epsilon_{2(m+1)+j} &= -\epsilon_{2m+j+1} + \mu\epsilon_{2m+j+1}^2 \\ &\to l - \mu l^2 + \mu\left(l - \mu l^2\right)^2 \\ &= l \end{aligned}$$

and hence

$$\begin{aligned} -\mu l^2 + \mu l^2 (1 - \mu l)^2 &= 0 \\ \mu l^2 \left(-1 + (1 - \mu l)^2\right) &= 0 \end{aligned}$$

so either $l = 0$ or $1 - \mu l = \pm 1$ so $l = (1 \pm 1)/\mu$. Hence $l = 0, 2/\mu$. But for $|\epsilon_0| < 2/\mu$ this cannot be obtained as the sequence $|\epsilon_n|$ is strictly decreasing. Hence $l = 0$, and the point is stable (though we expect convergence to be slow).

For the case of the logistic map, we have a smooth function, the polynomial

$$f = \lambda x (1 - x)$$

with derivative

$$f' = \lambda - 2\lambda x$$

so:

- $x = 0$ is stable if $|\lambda| < 1$ and unstable if $|\lambda| > 1$.
- $x = 1 - \frac{1}{\lambda}$ is stable if $|\lambda - 2\lambda + 2| = |2 - \lambda| < 1$. This is equivalent to $1 < \lambda < 3$. Similarly, for $\lambda < 1$ and $\lambda > 3$ this is unstable.

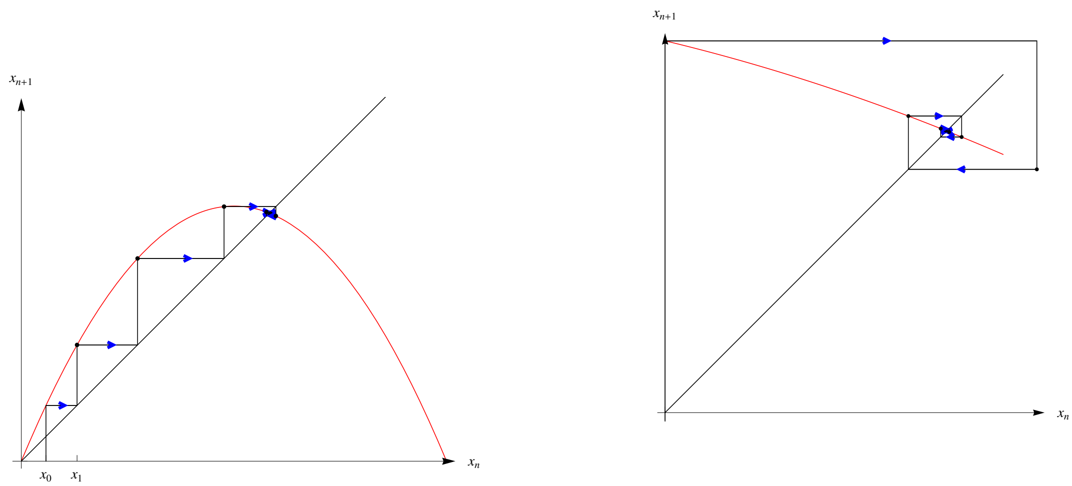
Note also that the ratio $\frac{\epsilon_{n+1}}{\epsilon_n} \approxeq f'(x) = 2 - \lambda$ which is positive for $\lambda < 2$ and negative for $\lambda > 2$. We have seen that x_n tends directly to the stable fixed point for $\lambda < 2$, as predicted above.

The oscillatory convergence can be seen for the case $2 < \lambda < 3$ in Figure 3.11.

Remark. For $\lambda = 3$, our analysis above for the case $f'(X) = -1$ indicates that there is a stable fixed point, since $f''(X) = -2\lambda < 0$.

Bifurcation The behaviour for $\lambda > 3$ gets rapidly more complicated, as there are no stable attracting points. It is important to realize that the maximum value obtained by x_n is at

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\lambda x (1 - x) &= \lambda - 2\lambda x = 0 \\ x &= \frac{1}{2} \end{aligned}$$



(a) Global behaviour (b) Detailed view of the local behaviour

Figure 3.11: Cobweb diagrams for the logistic map with $2 < \lambda < 3$

where it has value $\lambda/4$. Hence to keep the sequence x_n in $[0,1]$ we can have any $\lambda \in [0,4]$. Hence for a $\lambda \in [3,4]$ we have bounded behaviour *without* any stable fixed points.

The first interesting change is exhibited immediately after λ exceeds 3. The behaviour which can be seen is that the orbit on the cobweb diagram *expands* from the now unstable fixed point to become the limit cycle shown in Figure 3.12.

This is a cycle of period 2, so we are interested in the behaviour of $f^2(x) = f(f(x))$. Specifically, we want the fixed points of this map, as we are seeking cycles with $x_{n+2} = x_n$.

$$\begin{aligned}
f^2(x) &= x \\
\lambda f(x)\left[1 - f(x)\right] &= x \\
\lambda^2 x(1-x)\left[1 - \lambda x(1-x)\right] &= x \\
x\left[\lambda^2\left(1 - x(1+\lambda) + 2\lambda x^2 - \lambda x^3\right) - 1\right] &= 0 \\
-\lambda^3 x\left[x - \left(1 - \frac{1}{\lambda}\right)\right]\left[x^2 - \left(1 + \frac{1}{\lambda}\right) + \frac{1}{\lambda}\left(1 + \frac{1}{\lambda}\right)\right] &= 0
\end{aligned}$$

So the fixed points of order 2 are $x = 0$ and $1 - \frac{1}{\lambda}$ (because a fixed point of first order is trivially also a fixed point of second order, so we know this has to be a factor) and the roots of the final quadratic. These are given by

$$\begin{aligned}
x &= \frac{1 + \frac{1}{\lambda} \pm \sqrt{\left(1 + \frac{1}{\lambda}\right)^2 - \frac{4}{\lambda}\left(1 + \frac{1}{\lambda}\right)}}{2} \\
&= \frac{1 + \frac{1}{\lambda} \pm \sqrt{\frac{1}{\lambda^2}\left(\lambda^2 - 2\lambda - 3\right)}}{2} \\
&= \frac{\lambda + 1 \pm \sqrt{(\lambda - 3)(\lambda + 1)}}{2\lambda}
\end{aligned}$$

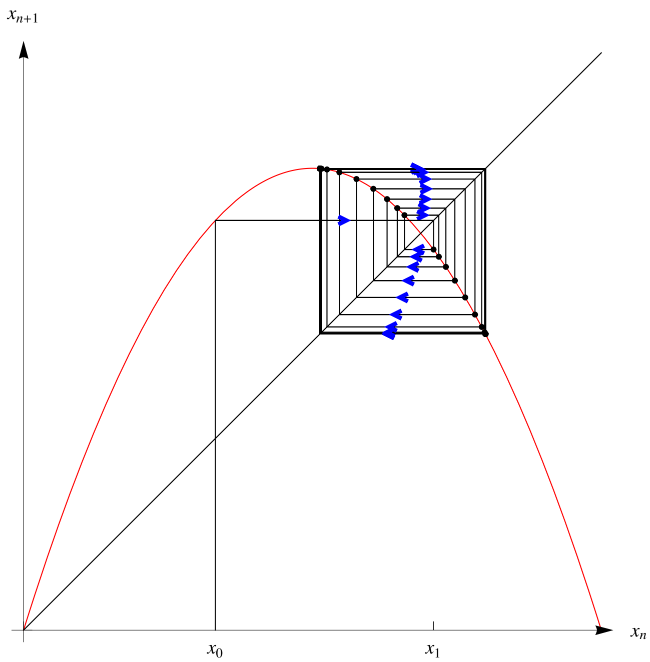


Figure 3.12: Cobweb diagram for the logistic map with $3 < \lambda < 1 + \sqrt{6}$

It can be seen that these are real for $\lambda \geq 3$, and distinct from the single fixed point for $\lambda > 3$.

Let $x_0 = 1 - \frac{1}{\lambda}$ be the unstable fixed point. Note also that

$$\frac{\mathrm{d}}{\mathrm{d}x} f\left(f\left(x\right)\right) = f'\left(x\right) f'\left(f\left(x\right)\right)$$

so the rate of change of f^2 at x_0 is given by

$$f'\left(x_0\right) f'\left(f\left(x_0\right)\right) \quad = \quad \left[f'\left(x_0\right)\right]^2$$

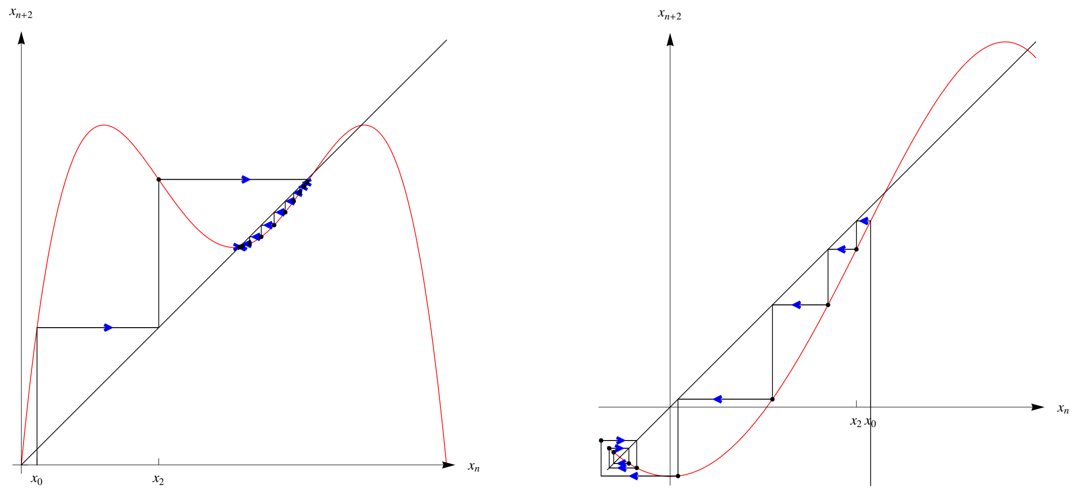
which as already noted is greater than 1 for all $\lambda > 3$. Similarly,

$$\begin{aligned} \left[\frac{\mathrm{d}^2}{\mathrm{d}x^2} f\left(f\left(x\right)\right)\right]_{x=x_0} &= f'\left(x_0\right)\left(1 + f'\left(x_0\right)\right) f''\left(x_0\right) \\ &< 0 \end{aligned}$$

as $f''\left(x_0\right) < 0$ because the gradient is falling (down past -1), and $f'\left(x_0\right) < -1$.

Therefore, locally around x_0, for λ slightly larger than 3, we expect the graph of $x_{n+2} = f^2\left(x_n\right)$ to cross the line $x_{n+2} = x_n$ steeply (and hence 'unstably') at x_0, but to dip back down again afterwards and dip up before, forming two complementary stable points of f^2, corresponding to the two points in the period 2 cycle. In fact, since $f''\left(x\right) < 0$ is negative for all x, and $f'\left(x\right)$ falls rapidly, this holds for any $\lambda > 3$.

The cases where the stable points have direct and oscillatory convergence correspond, as before, to the sign of $\mathrm{d}\left(f^2\right)/\mathrm{d}x$ at the two fixed points (i.e. the slope of the curve at the point of intersection), and examples of the behaviour of x_{2n} are shown in 3.13.



(a) Direct convergence of x_{2n} to the lower second order fixed point

(b) Oscillatory convergence of x_{2n} to the lower second order fixed point

Figure 3.13: Cobweb diagrams for the logistic map with $3 < \lambda < 1 + \sqrt{6}$

As one might expect, when the fixed points of f^2 themselves become unstable, the increased slope of the graph of f^4 at the old fixed points leads to the creation of two new nearby fixed points (which happens at $1 + \sqrt{6}$) in much the same way, which also move apart and eventually becomes unstable, and so on. This process of *period-doubling* or *bifurcation* continues, with the splits coming more and more rapidly.

One way of visualizing this process is to try to plot fixed points against λ. It rapidly becomes problematic to solve the polynomials of increasing order, so we usually do this fairly stochastically.

The technique often used is to pick, for each λ of interest, some starting point x_0, calculate and dispose of the first few iterations (say 100) and then plot its position thereafter, for say another 100 iterations. The idea is that the initial iterations allow the sequence to converge to one of its fixed points or cycles, and then we plot its progress through the cycle in which it is found. The result for $\lambda \in [0, 3.57]$ is shown in Figure 3.14.

The region $\lambda \in [3.4, 4]$ is shown in higher resolution in Figure 3.15.

Period doubling occurs at the points λ_i given for $i = 1, \cdots, 8$ by

$$(\lambda_i) = \begin{pmatrix} 3 \\ 3.449490\ldots \\ 3.544090\ldots \\ 3.564407\ldots \\ 3.568759\ldots \\ 3.569692\ldots \\ 3.569891\ldots \\ 3.569934\ldots \\ \vdots \end{pmatrix}$$

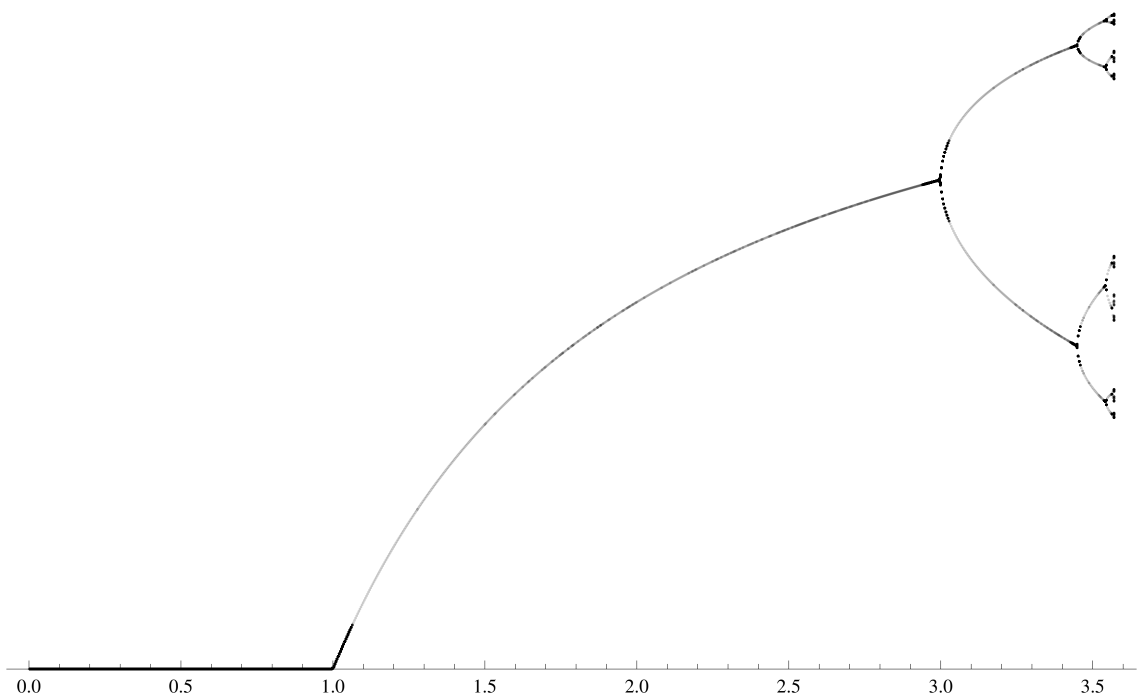


Figure 3.14: Bifurcation diagram for $\lambda < 3.57$

It appears that these values are converging rapidly to some *accumulation point* $\lambda_\infty = 3.56994567\ldots$ as can be verified by taking more terms. In fact, when the mathematician Feigenbaum observed this sequence, he guessed that it was approximately geometric, in the sense that the defect

$$\lambda_\infty - \lambda_n \approx c\delta^{-n}$$

for some constants c and δ.

Indeed, the ratio

$$\delta = \lim_{n\to\infty} \frac{\lambda_n - \lambda_{n-1}}{\lambda_{n+1} - \lambda_n}$$

exists, and has the value

$$\delta = 4.669201\ldots$$

and it is known as *Feignbaum's (first* or *delta) constant.* (c has a value of $2.637\ldots$)

Let $x^\star$ be the location of the maximum of f, which for the logistic map is $x^\star = 0.5$. For any n, find the λ such that $x^\star$ is in the 2^n-cycle. Now let d_n be the (signed) distance to $x^\star$ of the *closest other point* in the 2^n cycle - this is the distance between the two tines in this fork. Then the *Feigenbaum reduction parameter*

$$\alpha = \lim_{n\to\infty} \frac{d_n}{d_{n+1}}$$

also exists, with the value

$$\alpha = -2.502907\ldots$$

Amazingly, Feignbaum's constants δ and α are common to *any* period-doubling process arising in a one-dimensional system which has a single locally quadratic maximum. One may think of the δ constant as representing a universal scaling property in the λ-direction, and α in the x-direction. This shows that system is in fact approximately self-similar in the realm of period-doubling.

The region of period doubling forms only a small part of the diagram shown in Figure 3.15. A so-called *saddle-node bifurcation*, arising from a different shape in the graph of f^3 to what we studied for period-doubling bifurcations for f^2, occurs at $1+\sqrt{8} \approxeq 3.828$. This leads to a period 3 orbit at this point.

However, the behaviour is actually more complicated than simply increasingly complicated periodic structures - the orbit of the test point appears to fill out the entire interval. This is the onset of *chaos*, though it is highly structured, as the interested reader may find. (Of particular interest are the windows which open with stable oscillatory behaviour of period 3 or 7 etc. as a consequence of *mode locking* - it can be seen that these experience period doubling as well, before returning once more to chaos.)

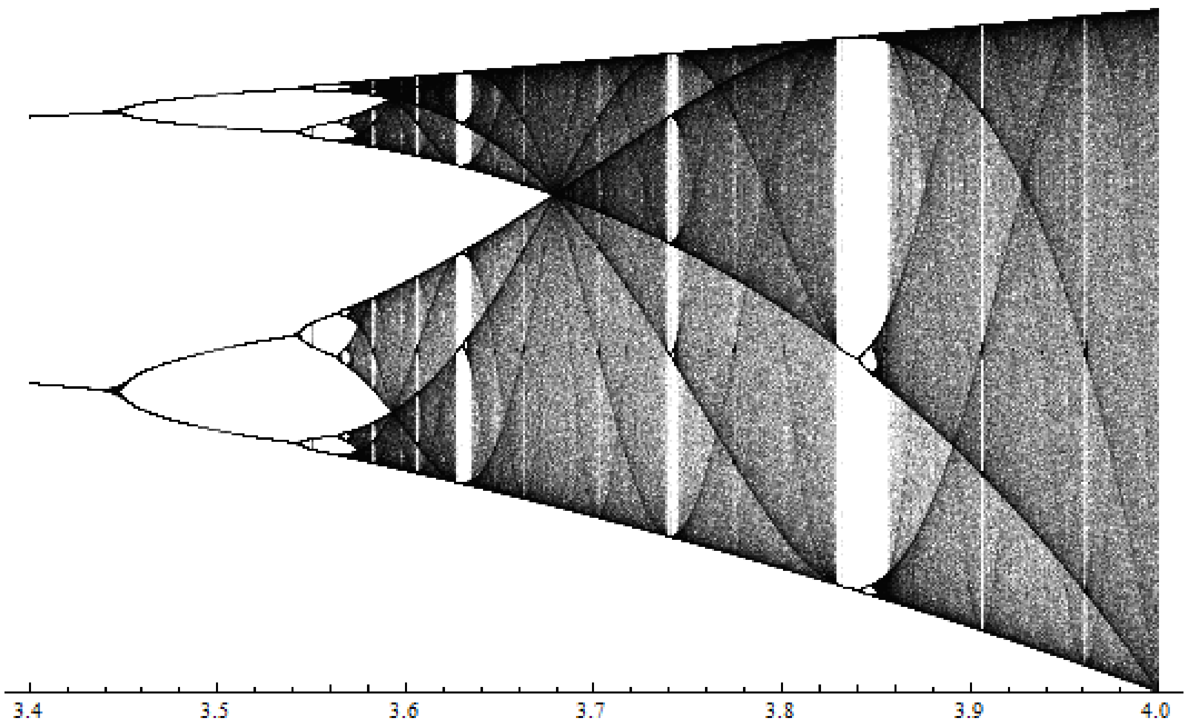


Figure 3.15: Bifurcation diagram for $\lambda \in [3.4, 4]$

3.7 * Existence and Uniqueness of Solutions

We will not go into any real detail in this course on the various results on existence and uniqueness of solutions to general differential equations. We will just state two key results key result and move on:

Theorem 3.21 (Peano existence theorem). *Consider an initial value problem*

$$y'(t) = f(t, y(t)), \quad y(t_0) = y_0, \quad t \in [t_0 - \epsilon, t_0 + \epsilon]$$

Suppose f is Lipschitz continuous in y and continuous in t. Then for some $\epsilon > 0$ there exists a solution $y(t)$ to the problem within the range $t \in [t_0 - \epsilon, t_0 + \epsilon]$.

Definition 3.22. A *Lipschitz continuous* function $g(x)$ on the real numbers is just a continuous function with the property that it 'never changes too fast' - so there is a K such that for any x and y,

$$|g(y) - g(x)| \leq K|y - x|$$

Remark. For example, if $f(t, y(t))$ is differentiable in y, then we are asking that $|\partial f/\partial y| \leq K$ at all points.

Theorem 3.23 (Picard-Lindelöf)**.** *If f is also Lipschitz continuous in y, then this solution is unique.*

You can check this result against the examples in section 3.5 - the ones which have a non-unique solution in any interval around an initial value do not satisfy these conditions. For a simple example:

Example 3.24. Solve $y' = |y|^{1/2}$ (defined, say, on the interval $[0, 1]$), given that $y(0) = 0$.

This is continuous in y and t, but we can easily come up with two totally different solutions: $y = 0$ and $y = x^2/4$. In fact, we can change to a function of the form $(x - C)^2/4$ wherever we like!

Indeed, note that

$$\frac{\partial}{\partial y}|y|^{1/2} = \frac{1}{2}|y|^{-1/2} \to \infty$$

as $y \to 0$, so this is not Lipschitz continuous near 0.

4 Second-Order Equations

In this section, we will move on to consider second-order systems, generalizing techniques where possible, and introducing new ones where appropriate. As before, we begin with a consideration of the simpler case of constant coefficients.

4.1 Constant Coefficients

The general form of such an equation is

$$ay'' + by' + cy = f(x)$$

with a, b, c constants.

To solve this, we follow the same basic two-stage procedure as before:

(i) Find complementary functions satisfying the homogeneous (unforced) version of the equation, $ay'' + by' + cy = 0$.

(ii) Find a particular integral satisfying the forced equation.

Remark. As before, this has strong analogies to what we technically think of as *affine* vector spaces, where there is some space of functions spanned by the eigenfunctions y_1 and y_2 found to be complementary functions so long as we move the origin to somewhere else, using the particular integral y_p. The ideas of vector spaces will recur in this section. Also, in section 4.3 we will use a geometrical description of a differential equation embedded in a vector space.

4.1.1 The complementary function

Recall that $e^{\lambda x}$ is an eigenfunction of $\mathrm{d}/\mathrm{d}x$, and consequently also of the repeated operator

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2} \equiv \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$$

though with a different eigenvalue, λ^2.

As a result, the complementary functions are of the form $y_c = e^{\lambda x}$, and hence $y_c' = \lambda y_c$, $y_c'' = \lambda^2 y_c$ and so

$$\boxed{a\lambda^2 + b\lambda + c = 0}$$

is the *characteristic equation* for the eigenvalue λ. This has two solutions λ_1 and λ_2 which *may be equal.*

So we have two solutions to the differential equation, namely

$$y_1 = e^{\lambda_1 x} \quad \text{and} \quad y_2 = e^{\lambda_2 x}$$

Of course, in the case $\lambda_1 = \lambda_2$, these are the same function, and we only have one degree of freedom in our attempted solution of $Ay_1 + By_2$ - clearly, we are missing something. Otherwise, though, we have the following result, which we will not prove:

Theorem 4.1. *If the eigenvalues λ_1 and λ_2 are distinct, then y_1 and y_2 are* linearly independent *and* complete *- that is, they form a* basis *of the space of solutions of the homogeneous equation.*

Remark. Again note that this is expressed in the language of vector spaces. All of these results are straightforwardly generalized to equations of higher order in the natural way.

The most general complementary function in the case of distinct roots is therefore

$$\boxed{y_c = Ae^{\lambda_1 x} + Be^{\lambda_2 x}}$$

Example 4.2. Solve $2y'' + 3y' + 1 = 0$.

Looking for solutions of the form $e^{\lambda x}$ we find

$$\begin{aligned} 2\lambda^2 + 3\lambda + 1 &= 0 \\ \lambda &= \frac{-3 \pm \sqrt{3^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} \\ &= \frac{-3 \pm 1}{4} \\ &= -1, -\frac{1}{2} \end{aligned}$$

Therefore,

$$y = e^{-x}, e^{-x/2}$$

are both solutions of this form, and the general solution is

$$y = Ae^{-x} + Be^{-x/2}$$

In the case that λ_1 and λ_2 have imaginary parts, we can rewrite the solution in a more comprehensible form by using Euler's formula:

Example 4.3. Solve $y'' + 2y' + 5 = 0$.

This time, we have

$$\begin{aligned} \lambda &= \frac{-2 \pm \sqrt{2^2 - 4 \cdot 5}}{2} \\ &= -1 \pm \sqrt{-4} \\ &= -1 \pm 2i \end{aligned}$$

So we can write

$$\begin{aligned} y &= Ae^{2ix-x} + Be^{-2ix-x} \\ &= e^{-x}\left[Ae^{2ix} + Be^{-2ix}\right] \\ &= e^{-x}\left[(A+B)\cos 2x + (A-B)\, i\sin 2x\right] \\ &= e^{-x}\left[C\cos 2x + D\sin 2x\right] \end{aligned}$$

for some in general complex $C, D \in \mathbb{C}$. However, clearly they will be real for real initial conditions.

Remark. In this last line we have found two functions $e^{-x}\cos 2x$ and $e^{-x}\sin 2x$ which *are* linearly independent and therefore form a basis of the unforced equation given. But they are *not* eigenfunctions of the individual differential operator $\mathrm{d}/\mathrm{d}x$, only of the given left-hand side.

Finally, let us see an example of an equation where the two roots coincide:

Example 4.4 (Degeneracy)**.** Solve $y'' - 6y' + 9y = 0$.

In this case we find

$$\begin{aligned} \lambda^2 - 6\lambda + 9 &= 0 \\ (\lambda - 3)^2 &= 0 \end{aligned}$$

so that $\lambda_1 = \lambda_2 = 3$ and there is only one solution. As a result, it is clear that e^{3x} and e^{3x} are not linearly independent, and as a result they do not form a basis (because we assume the space is two dimensional, so there must be two basis functions).

4.1.2 Detuning degenerate equations

The first method we will use to solve this equation is the same as we employed in Example 3.10, which had an equation forced by a root of the characteristic equation - we form a family of slightly different version of the same equation parametrized by some small ϵ and then let $\epsilon \to 0$.

Example 4.5. Consider the equations given by

$$y'' - 6y' + \left(9 - \epsilon^2\right) y = 0$$

This way of inserting the parameter is simple the easiest to deal with in the general solution of the equation for $\epsilon \neq 0$: we try $y = e^{\lambda x}$ and find $\lambda = 3 \pm \epsilon$.

So

$$\begin{aligned} y_c &= Ae^{(3+\epsilon)x} + Be^{(3-\epsilon)x} \\ &= e^{3x}\left(Ae^{\epsilon x} + Be^{-\epsilon x}\right) \end{aligned}$$

Now as before, we take a Taylor expansion about $\epsilon = 0$ (not x) and find

$$\begin{aligned} y_c &= e^{3x}\left[A\left(1+\epsilon x+O\left(\epsilon^2\right)\right)+B\left(1-\epsilon x+O\left(\epsilon^2\right)\right)\right] \\ &= e^{3x}\left[(A+B)+\epsilon x\,(A-B)+O\left(A\epsilon^2, B\epsilon^2\right)\right] \end{aligned}$$

The important idea at this point is to choose the particular solutions (values of A and B) that we are interested in for each ϵ. In this case, it seems like a good idea to have

$$\begin{aligned} A+B &= \alpha \\ \epsilon\,(A-B) &= \beta \end{aligned}$$

so that α, β are the two new parameters that we are allowed to choose.

Then

$$\begin{aligned} A &= \frac{1}{2}\left(\alpha+\frac{\beta}{\epsilon}\right)=O\left(\frac{1}{\epsilon}\right) \quad \text{as } \epsilon \to 0 \\ B &= \frac{1}{2}\left(\alpha-\frac{\beta}{\epsilon}\right)=O\left(\frac{1}{\epsilon}\right) \quad \text{as } \epsilon \to 0 \end{aligned}$$

so that terms like $A\epsilon^2 = O(\epsilon)$, and hence the solution

$$\begin{aligned} y_c &= e^{3x}\left(\alpha+\beta x+O(\epsilon)\right) \\ &\to e^{3x}\left(\alpha+\beta x\right) \quad \text{as } \epsilon \to 0 \end{aligned}$$

So we are inclined to guess that this two-parameter function is in fact a solution in the case $\epsilon = 0$, though this is not quite a formal proof of the fact. Indeed, if we check it, we readily see that both e^{3x} and xe^{3x} are solutions of the equation.

Recall from section 3.3.4 on resonant forcing terms that if $e^{\lambda x}$ solves a similar first-order equation, then the general form of a particular solution with forcing term $f(x) = ce^{\lambda x}$ is

$$y_p = dxe^{\lambda x}$$

It seems we have a similar result here: perhaps all repeated roots are accompanied by an exponential solution with an added factor of x in the complementary function. Indeed, this is always the case (see section 6.3).

4.1.3 Reduction of order

When we have a polynomial equation of degree n, and we know a factor $(x - \lambda_1)$, we *factor out* this term to get a simpler equation of degree $n - 1$. You might have wondered if we could do something similar in the analogous differential equation when we know one solution.

In turns out that for the analogous case of the homogeneous, linear differential equation we can do exactly that. It is called the method of reduction of order.

Assume that y_1 is a complementary function solving

$$ay'' + by' + c = 0$$

Then we will look for another solution y_2 given by

$$y_2(x) = v(x)\, y_1(x)$$

We ignore for the moment complications like the fact that $y_1(x_0) = 0$ would guarantee $y_2(x) = 0$.

Now since y_2 is another solution to this equation, we have

$$\begin{aligned} 0 &= a(vy_1)'' + b(vy_1)' + c(vy_1) \\ &= (ay_1'' + by_1' + cy_1)\, v + (2ay_1' + by_1)\, v' + (ay_1)\, v'' \\ &= (2ay_1' + by_1)\, v' + (ay_1)\, v'' \\ &= \left(2a\frac{y_1'}{y_1} + b\right) v' + av'' \end{aligned}$$

Now since v does not appear in this equation - you can see it must always cancel, even for higher order equations - this is actually a *first-order equation for v'*. So if we solve a first-order equation, then we determine v', and hence v up to a constant additive factor, which makes sense because $(v + C)\, y_1$ is obviously a solution if vy_1 is.

If $y_1 = e^{\lambda x}$ then we have

$$\begin{aligned} (2a\lambda + b)\, v' + av'' &= \frac{\mathrm{d}}{\mathrm{d}x}\left[(2a\lambda + b)\, v + av'\right] = 0 \\ av' + (2a\lambda + b)\, v &= C \end{aligned}$$

Then if λ is a repeated root, we have $\lambda = -b/2a$ so this gives

$$\begin{aligned} av' &= C \\ v &= Dx \quad (+E) \end{aligned}$$

which gives the general solution

$$y_2 = (Dx + E)\, e^{\lambda x}$$

This technique can actually be applied more generally when we do not have constant coefficients.

4.2 Particular Integrals and Physical Systems

4.2.1 Resonance

Most of the ideas about particular integrals carry straight over from the first-order case. The following table gives the rules of thumb for linear equations with constant coefficients:

Forcing term $f(x)$	Guess for particular solution $y_p(x)$
e^{mx}	Ae^{mx}
$\sin kx$, $\cos kx$	$A\sin kx + B\cos kx$
$p(x) = p_n x^n + \cdots + p_1 x + p_0$	$q(x) = q_n x^n + \cdots + q_1 x + q_0$

Remember that for linear equations, we can consider forcing terms one at a time, superposing (i.e. adding together) the resulting solutions.

Again, we must consider what happens when we get forcing proportional to a complementary function. From past experience, we expect to get an additional factor of x multiplying any such eigenfunction, and this is roughly what happens. We already know that we can detune equations to make educated guesses at the form of solutions, and we will see in section 4.4 one new way of obtaining these results without guesswork, but for now let us simply note that the following approach works:

Example 4.6. Solve $y'' - 2y' + y = e^x + xe^x$.

In this case, we know the complementary functions of the equation can be written as e^x and xe^x, so both forcing terms are proportional to eigenfunctions.

Let us try and deal with the e^x term first - clearly, we know that a guess of the form ae^x will not work, but also that axe^x will not either. So try $y_p = ax^2e^x$ - it turns out that this does indeed work:

$$y_p'' - 2y_p' + y_p = \cdots = 2ae^x$$

Similarly, it turns out that $y_p = ax^3e^x$ gives

$$y_p'' - 2y_p' + y_p = \cdots = 6axe^x$$

The most interesting case, however, is when a system which would normally oscillate at some frequency ω_0 is forced at that same frequency; this results in a phenomenon called *resonance*. Because of the highly physical nature of this problem, we will consider functions $y(t)$ varying in time with derivatives $\dot{y}$ and $\ddot{y}$.

Consider the equation

$$\ddot{y} + \omega_0^2 y = 0$$

This has the simple solution

$$y = A\sin\omega_0 t + B\cos\omega_0 t$$

The question is: what happens if we add a forcing term of $\sin\omega_0 t$?

We could leap straight in and guess that there is a particular solution of the form containing a term like $t\sin\omega_0 t$, and possibly also $t\cos\omega_0 t$, but since it is useful to see what happens when the system is forced at frequencies not *quite* equal to the natural frequency ω_0, we will proceed once more by the method of detuning.

Our equation has the form

$$\ddot{y} + \omega_0^2 y = \sin\omega t \quad \omega \neq \omega_0$$

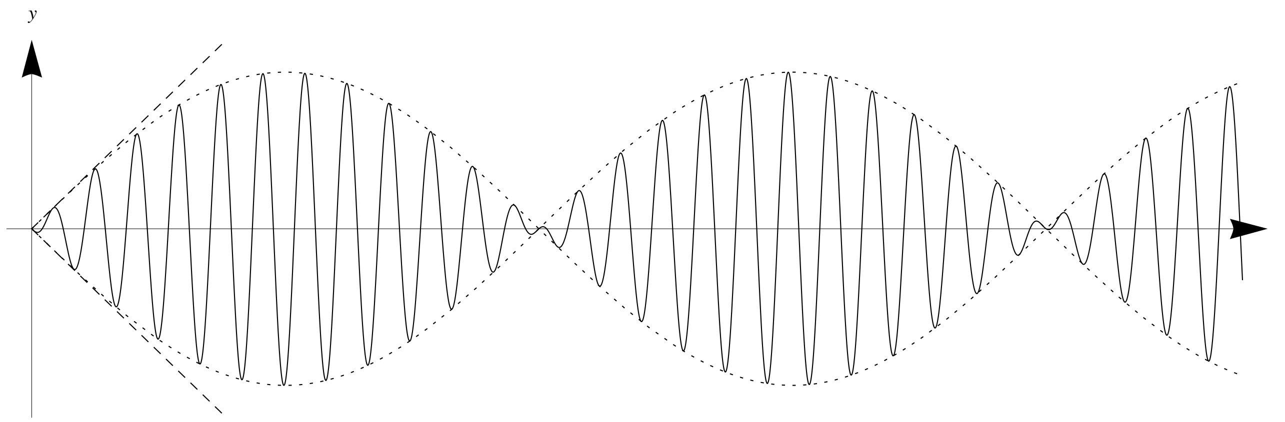


Figure 4.1: Diagram of near-resonating sinusoidal oscillations, displaying beating

Our guess for a particular solution, then, is accordingly one with $\sin\omega t$ and $\cos\omega t$. In fact, we anticipate that $\cos\omega t$ will not appear, since there is no $\dot{y}$ term in the equation (you can easily check this).

But since we are interested in adjusting the particular solution to look more and more like a solution to the homogeneous equation, we can actually subtract the most similar such solution from the particular solution. Then our guess for the particular solution is

$$\begin{aligned} y_p &= C\left[\sin\omega t - \sin\omega_0 t\right] \\ \ddot{y}_p &= C\left[-\omega^2\sin\omega t + \omega_0^2\sin\omega_0 t\right] \end{aligned}$$

and we can therefore get

$$C\left(\omega_0^2 - \omega^2\right) = 1$$

which leaves us with

$$y_p = \frac{\sin\omega t - \sin\omega_0 t}{\omega_0^2 - \omega^2}$$

We want to rewrite this in terms of $\Delta\omega = \omega_0 - \omega$, so we use one of the trigonometric identities for sums of sine functions to get

$$\begin{aligned} y_p &= \frac{2}{(\omega_0 + \omega)\,\Delta\omega}\left[\cos\frac{\omega t + \omega_0 t}{2}\sin\frac{\omega t - \omega_0 t}{2}\right] \\ &= \frac{-1}{\left(\omega_0 - \frac{\Delta\omega}{2}\right)\Delta\omega}\left[\cos\left(\omega_0 - \frac{\Delta\omega}{2}\right)t\cdot\sin\frac{\Delta\omega}{2}t\right] \end{aligned}$$

This way of expressing the particular solution we have found neatly summarizes most of its properties.

- The magnitude of the sinusoidal functions is always bounded by 1, so the maximum magnitude of this solution is roughly $1/\left(\omega_0\Delta\omega\right)$, which grows rapidly as $\Delta\omega \to 0$.
- There is an underlying sinusoidal oscillation of frequency $(\omega_0 - \Delta\omega/2) \approxeq \omega_0$, just as in the complementary function.

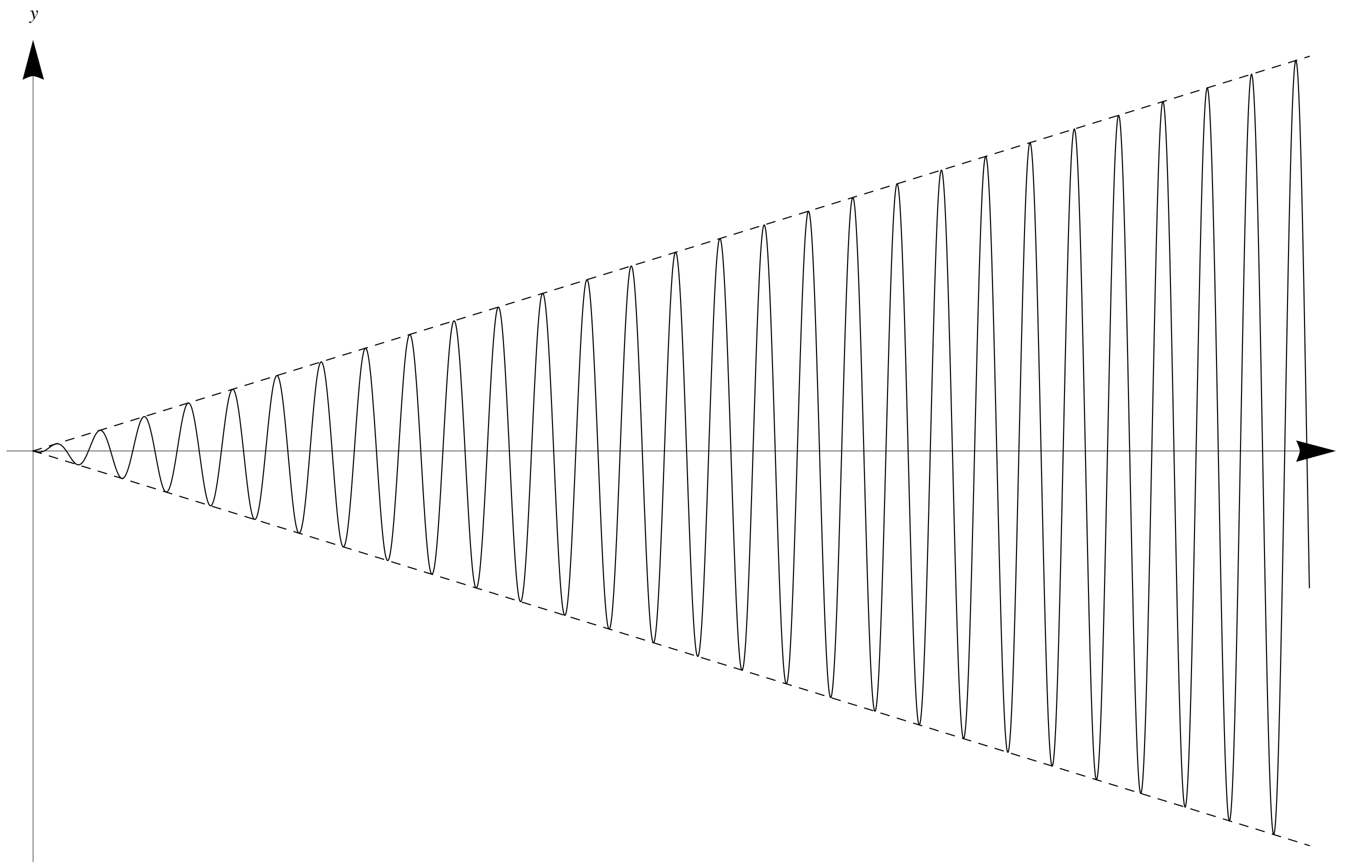


Figure 4.2: Diagram of eventual linear growth

- There is an envelope of a sinusoidal function with frequency $\Delta\omega/2 \to 0$, and hence period $2/\Delta\omega \to \infty$.

As a result, when we force the system at close to its natural frequency, we observe what is termed *beating*, as shown in Figure 4.1. This modulation of amplitude is clearly audible in some audio setups.

We can also see what happens as $\Delta\omega \to 0$ - the wavelength of the envelope function tends to infinity, and its shape tends towards the fixed linear cone shown as dashed lines in the figure - exactly as expected. This limiting case is shown in Figure 4.2.

Mathematically speaking, as $\Delta\omega \to 0$ we get

$$\begin{aligned} y_p(t) &\to -\frac{1}{\omega_0}(\cos\omega_0 t)\cdot\left(\frac{\frac{\Delta\omega}{2}t}{\Delta\omega}\right) \\ &= -\frac{t}{2\omega_0}\cos\omega_0 t \end{aligned}$$

Remark. Notice that the oscillation here is generally like $\cos\omega_0 t$ - this is a characteristic phenomenon in forced oscillators: they often oscillate out of phase with the driving force.

4.2.2 Damped oscillators

In the last subsection, we saw how physical systems which would naturally oscillate indefinitely can behave. Often, however, physical systems have some sort of damping force which restricts this behaviour, causing the system to eventually come to rest in the absence of a driving force. The force leading to oscillation is the so-called *restoring force*, which always 'points towards the equilibrium solution'.

Let's see what we mean by this. Imagine a mass M attached to a spring, which exerts a force proportional to the extension from its natural length. Write x for the position of the mass, choosing the origin $x = 0$ to be the equilibrium so that the spring exerts a force $F_s = -kx$ on the mass (k is the *spring constant*). We can force the spring with some force $F(t)$ if we like.

At the moment, the equation of motion of the system, given by Newton's second law, is

$$\begin{aligned} M\ddot{x} &= -kx + F(t) \\ M\ddot{x} + kx &= F(t) \end{aligned}$$

which is, as we know, a forced simple harmonic oscillator.

Now suppose we restrict the motion by adding a damping mechanism, like a shock absorber in a car. Let us imagine, for a concrete example, that the motion forces a piston to move through oil, exerting a force of magnitude $|l\dot{x}|$ opposing the motion due to the oil's viscosity.

The new equation is

$$\boxed{\ddot{x} + \frac{l}{M}\dot{x} + \frac{k}{M}x = \frac{F(t)}{M}}$$

On physical grounds, for the unforced version of this equation, we expect behaviour roughly similar to the harmonic oscillator, with some sort of decay over time (for example, by considering energy lost doing work on the oil). In fact, we could solve this directly using the techniques available to us.

However, it is useful to parameterize the system more concisely. We can reduce the number of free parameters by first rescaling time via

$$t = \sqrt{\frac{M}{k}}\tau$$

to eliminate the term multiplying x, and then we have

$$\boxed{\frac{\mathrm{d}^2x}{\mathrm{d}\tau^2} + 2\kappa\frac{\mathrm{d}x}{\mathrm{d}\tau} + x = f(\tau)}$$

where

$$\kappa = \frac{l}{2\sqrt{kM}} \quad \text{and} \quad f = \frac{F}{k}$$

This way of expressing the equation of motion shows that the fundamental characteristics of the solution are controlled entirely by the parameter κ (the system then needs to be rescaled in time to account for the $t \to \tau$ map).

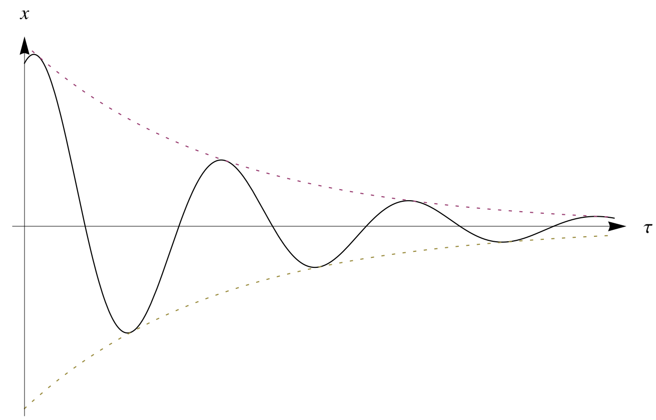


Figure 4.3: Lightly damped oscillator

We will first restrict our analysis to the case of an undamped system, $f = F = 0$.

$$x'' + 2\kappa x' + x = 0$$

Trying solutions of the form $e^{\lambda\tau}$ gives us the characteristic equation

$$\begin{aligned}
\lambda^2 + 2\kappa\lambda + 1 &= 0 \\
\lambda &= -\kappa \pm \sqrt{\kappa^2 - 1} \\
&= \lambda_1, \lambda_2
\end{aligned}$$

This naturally splits into three different cases, which we will analyze separately.

Lightly damped oscillator In the case $\kappa < 1$, we have a solution of the form

$$x = e^{-\kappa t}\left(A\sin\left(\sqrt{1-\kappa^2}\tau\right) + B\cos\left(\sqrt{1-\kappa^2}\tau\right)\right)$$

We call this a *damped oscillator*: we can see it oscillates at a constant frequency, but with an exponentially decaying amplitude. An example is shown in Figure 4.3.

The period of oscillation is

$$\begin{aligned}
T_\tau &= \frac{2\pi}{\sqrt{1-\kappa^2}} \\
T &= \frac{2\pi}{\sqrt{1-\frac{l^2}{4kM}}} \cdot \sqrt{\frac{M}{k}} \\
&= \frac{4\pi M}{\sqrt{4kM - l^2}}
\end{aligned}$$

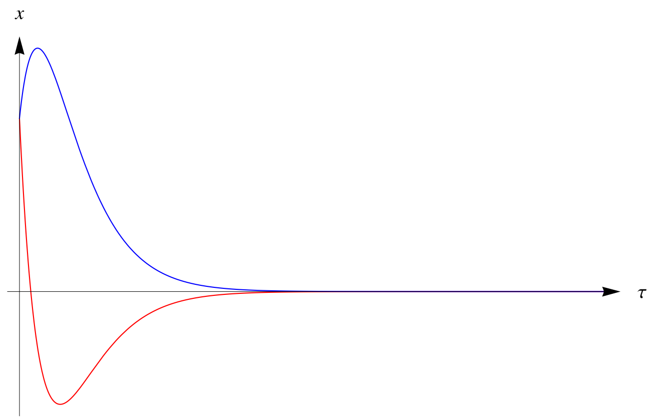


Figure 4.4: Critically damped oscillator

and the decay time, or time taken to reach some specific fraction of the original amplitude, is

$$O\left(\frac{1}{\kappa}\right)$$

with a characteristic time (or *e-folding time*, the time needed to reach an amplitude equal to e^{-1} times the original amplitude) of

$$\begin{aligned} \tau &= \frac{1}{\kappa} \\ t &= \sqrt{\frac{M}{k}} \cdot \frac{1}{\kappa} \\ &= \sqrt{\frac{M}{k}} \cdot \frac{2\sqrt{kM}}{l} \\ &= \frac{2M}{l} \end{aligned}$$

(A related quantity, the Q-factor, is given by $\frac{1}{2\kappa}$.)

Note that as $\kappa \to 1$, the oscillation period tends to ∞. This gives us some indication of what to expect in the next case.

Critical damping Here, the solution is

$$x = (A + B\tau)\, e^{-\kappa\tau}$$

where we can calculate A and B from the initial conditions:

$$\begin{aligned} x(0) &= A \\ x'(0) &= B - \kappa A \end{aligned}$$

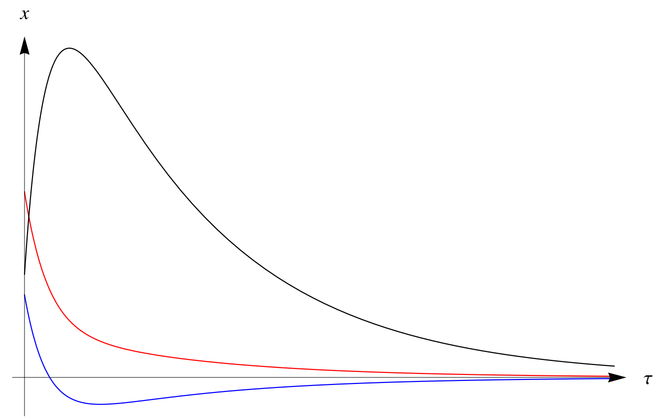


Figure 4.5: Over-damped oscillator

so

$$\begin{aligned} A &= x(0) \\ B &= x'(0) + \kappa x(0) \end{aligned}$$

where the derivative $x'(0)$ is with respect to τ (not t).

This solution decays to 0 most rapidly, for any value of κ.

Note that it displays two main kinds of behaviour, as shown in Figure 4.4 - assuming without loss of generality that $x(0) > 0$, either $B \geq 0$ and the system decays essentially exponentially to 0 (with an initial deceleration if $x'(0) > 0$), or $x'(0) < 0$; or $B < 0$, and the system passes through the origin once before gradually decaying coming to rest from the negative direction.

The decay time remains in terms of τ is still $O(1/\kappa)$, and also the time to the peak is $O(1/\kappa)$.

Over damping The final case has a general solution

$$\begin{aligned} x &= Ae^{\lambda_1 \tau} + Be^{\lambda_2 \tau} \\ &= e^{-\kappa\tau}\left[Ae^{\sqrt{\kappa^2-1}\tau} + Be^{-\sqrt{\kappa^2-1}\tau}\right] \end{aligned}$$

where $\lambda_1, \lambda_2 < 0$.

Here, the decay time is $O(1/\lambda_1)$ and the time to the peak is $O(1/\lambda_2)$. Note again, as shown in Figure 4.5, that there are various possible types of behaviour, depending on the initial conditions.

Forced systems In a forced system like this, the complementary function determines the *short-term, transient response*, while the particular integral determines the *long-term asymptotic behaviour* of the system, provided that the forcing term is not dying away over time.

Example 4.7. $\ddot{x} + 2\kappa\dot{x} + x = \sin\tau$, $\kappa > 0$.

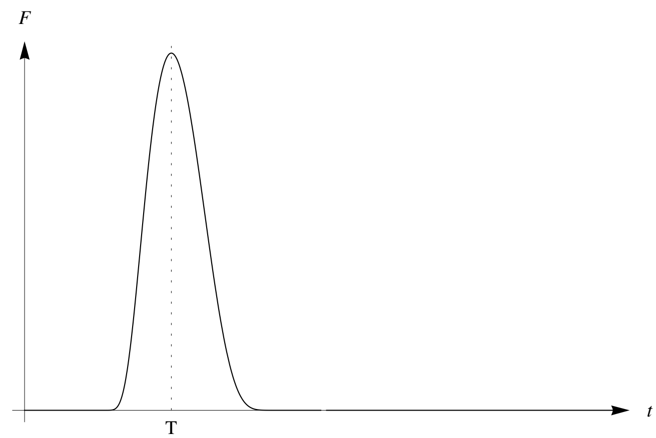


Figure 4.6: An example of an approximate impulse force

We want to try to find a particular solution. Trying $x = C \sin \tau + D \cos \tau$ gives $C = 0$ and $D = -1/2\kappa$.

Hence

$$x = Ae^{\lambda_1 \tau} + Be^{\lambda_2 \tau} - \frac{1}{2\kappa} \cos \tau$$

and we know that the complementary function will tend to 0 over time, so we can say that *asymptotically*, x tends to the particular integral:

$$x \sim -\frac{1}{2\kappa} \cos \tau \quad \text{as } \tau \to \infty$$

(The notation $f \sim g$ means $f/g \to 1$ in the stated limit.)

In general, the response to $e^{i\omega\tau}$ is $Ae^{i\omega\tau}$ for some $A \in \mathbb{C}$. Writing $A = re^{i\theta}$, the response is $re^{i(\omega\tau+\theta)}$, giving rise to a phase-shift of θ in the response. This shift depends on κ and the frequency ω.

4.2.3 Impulse and point forces

So far, we have dealt entirely with continuous (and indeed usually *smooth*, or infinitely differentiable) forcing terms f. Indeed, physically, we expect physical laws to behave mostly in a continuous manner - positions do not instantly change. However, at least at the macroscopic scale, some events seem to involve instantaneous or perfectly localized changes, and it may be a simplifying assumption to assume this, rather than take account of all the tiny, short-lived changes that add up roughly to a very simple model. We may find, then, that velocities or accelerations can be discontinuous in our new model.

Consider, as an example, a ball falling to the ground and bouncing. The force due to hitting the ground, $F(t)$, will be entirely contained in $[t_1, t_2]$, where the ball first makes contact with the ground at t_1 and leaves it totally at t_2. The size of this interval will be small compared to the time spent in the air for a reasonably rigid or elastic ball, and we can pick a representative point in this interval, say T, such that the force is almost entirely concentrated at within some tiny interval $[T-\epsilon, T+\epsilon]$ of length $O(\epsilon)$.

The form of $F(t)$ in this case would be very difficult to calculate even numerically, since the ball would in practice begin vibrating internally, and elastically changing shape, so we would need to know a lot of information about the ball's internal structure, in order to calculate its vibrational modes, and so on. But the approximation shown in Figure 4.6 is roughly what we are going to assume to be the general rule, since it does not really make any difference what the exact form of F is for our purposes.

It is then more convenient for the purposes of our analytic approximation to assume that the collision occurs instantaneously at time $t = T$ - we are essentially taking the physical limit $\epsilon \to 0$. Mathematically, Newton's second law states (assuming locally constant gravity) that

$$m\ddot{x} = F(t) - mg$$

so integrating over the small range $[T-\epsilon, T+\epsilon]$ we find

$$\begin{aligned}\int_{T-\epsilon}^{T+\epsilon} m\ddot{x}\mathrm{d}t &= \int_{T-\epsilon}^{T+\epsilon} F(t)\,\mathrm{d}t - \int_{T-\epsilon}^{T+\epsilon} mg\mathrm{d}t \\ \left[m\frac{\mathrm{d}x}{\mathrm{d}t}\right]_{T-\epsilon}^{T+\epsilon} &= I - 2mg\epsilon\end{aligned}$$

where $I = \int F\mathrm{d}t$ is defined to be the *impulse* due to F, the area under the force-time curve. This transfer of momentum is the only macroscopic property of F which is relevant to the variable x.

Hence, for small ϵ,

$$\Delta\text{momentum} = \left[m\frac{\mathrm{d}x}{\mathrm{d}t}\right]_{T-\epsilon}^{T+\epsilon} \approxeq I$$

So the only feature of $F(t;\epsilon)$ - $F(t)$ is parameterized by ϵ to give a family of increasingly localized forces as $\epsilon \to 0$ - which we are considering is its *time-integral*.

We are considering a family of functions $D(t,\epsilon)$ such that

$$\lim_{\epsilon\to 0} D(t,\epsilon) = 0 \quad \text{for all } t \neq 0$$

and

$$\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} D(t;\epsilon)\,\mathrm{d}t = 1$$

These two properties define, in the limit $\epsilon \to 0$, a *distribution*, which is not a true function, but something defined almost entirely in terms of its properties when multiplied by a function and integrated over some suitable range.

Remark. An example of such a family of functions D is given by

$$D(t;\epsilon) = \frac{1}{\epsilon\sqrt{\pi}}e^{-\frac{t^2}{\epsilon^2}}$$

and shown in Figure 4.7.

It is clear that as $\epsilon \to 0$, the central point $D(0,\epsilon) \to \infty$, which is why there is no suitable function

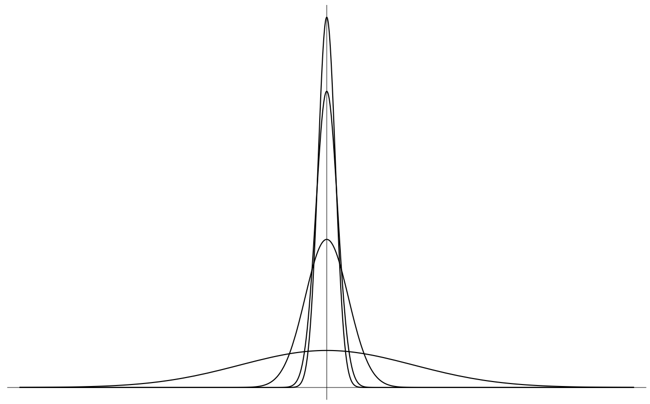


Figure 4.7: A family of approximate impulses

with the properties of the limit:

$$\lim_{\epsilon\to 0} D(t;\epsilon)$$

is not defined.

However, in practice we think of this limit as defining a function:

Definition 4.8. The *Dirac delta function* is the *generalized function* or *distribution* $\delta(x)$ satisfying

$$\int_a^b f(x)\,\delta(x-c)\,\mathrm{d}x = \begin{cases} f(c) & \text{if } c\in(a,b) \\ 0 & \text{if } c\notin[a,b] \end{cases}$$

for any suitable f (in particular, we require f to be continuous at 0). (We leave the problematic cases of $c=a$ and $c=b$ aside here.)

Remark. One way of expressing this formally is to say that δ is a *linear functional* on some space of functions, so that $\delta[af(x)+bg(x)] = af(0)+bg(0)$.

Note that, for example,

$$\begin{aligned} \lim_{\epsilon\to 0}\int_{-\infty}^{\infty} f(x)\,D(x;\epsilon)\,\mathrm{d}x &= f(0)\lim_{\epsilon\to 0}\int_{-\infty}^{\infty} D(x;\epsilon)\,\mathrm{d}x \\ &= f(0) \end{aligned}$$

for continuous f.

This 'function' (for our purposes here, we shall act as if it was a typical function) is convenient for representing and doing calculations with impulses and point forces:

Example 4.9. For the bouncing ball considered above, we can write

$$m\ddot{x} = -mg + I\delta(t-T)$$

with, say, $x(0)=x_0$ and $\dot{x}(0)=0$. Then we have

(i) For $0 \leq t < T$, we have $m\ddot{x} = -mg$ so

$$x = -g\frac{t^2}{2} + At + B$$

The initial conditions give $B = x_0$ and $A = 0$, and thus

$$x(t) = -\frac{1}{2}gt^2 + x_0$$

(ii) For $T < t < \infty$, we also have $m\ddot{x} = -mg$, so again

$$x = -\frac{1}{2}gt^2 + Ct + D$$

This time, we apparently lack initial conditions to give us the values of C and D.

(iii) To obtain a solution, we first make the (physically based) assumption that $x(t)$ is continuous at $t = T$. As t grows to T from below, we have

$$\lim_{t \to T^-} x(t) = -\frac{1}{2}gT^2 + x_0$$

so we expect this to be the value of $x(T)$. Then by the assumption, we have

$$\begin{aligned} -\frac{1}{2}gT^2 + CT + D &= -\frac{1}{2}gT^2 + x_0 \\ CT + D &= x_0 \end{aligned}$$

(iv) We still need more information to calculate C and D. We can gain some integrating the differential equation over a small interval around the time of the impulse:

$$\begin{aligned} \int_{T-\epsilon}^{T+\epsilon} m\ddot{x}\mathrm{d}t + \int_{T-\epsilon}^{T+\epsilon} mg\mathrm{d}t &= \int_{T-\epsilon}^{T+\epsilon} I\delta(t-T)\,\mathrm{d}t \\ \Delta[m\dot{x}] + 2mg\epsilon &= I \end{aligned}$$

so taking the limit $\epsilon \to 0$ we can deduce the change in $\dot{x}$ at time T

$$\Delta\dot{x} = \frac{I}{m}$$

Now we can write down the velocities before and after to obtain another equation:

$$\begin{aligned} [-gT] + \frac{I}{m} &= [-gT + C] \\ C &= \frac{I}{m} \end{aligned}$$

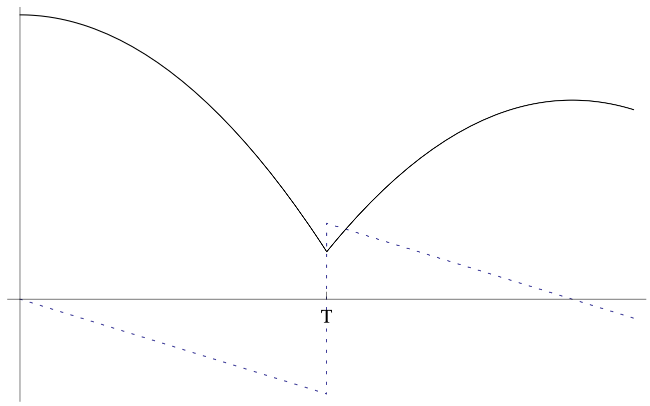


Figure 4.8: The solution to the bouncing ball problem, with the velocity superimposed as a dotted line.

(v) Hence

$$\begin{aligned} x(t) &= \begin{cases} -\frac{1}{2}gt^2 + x_0 & \text{for } t \leq T \\ -\frac{1}{2}gt^2 + \frac{I}{m}t + x_0 - \frac{I}{m}T & \text{for } t > T \end{cases} \\ &= -\frac{1}{2}gt^2 + x_0 + \begin{cases} 0 & \text{for } t \leq T \\ \frac{I}{m}(t-T) & \text{for } t > T \end{cases} \end{aligned}$$

as shown in Figure 4.8.

Remark. Suppose that the solution $x(t)$ had been discontinuous at $t = T$ - then $x'(t)$ would have a δ-like singularity at $t = T$, and $x''(t)$ would behave 'even more singularly', like the highly irregular function 'δ'', at $t = T$. The equation $mx'' + mg = I\delta(t-T)$ would then not be satisfied, as can be seen (for example) by integrating it and noting that the left-hand side has a δ-type singularity, but the right-hand side is simply a discontinuous step.

The result of integrating the δ function is clearly always a discontinuous step.

Definition 4.10. The *Heaviside step function* is defined by

$$H(x) \equiv \int_{-\infty}^{x} \delta(t)\,\mathrm{d}t = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \\ \text{undefined} & \text{at } x = 0 \end{cases}$$

We can write '$\frac{\mathrm{d}H}{\mathrm{d}x} = \delta(x)$' so long as we are careful only to use this relationship inside integrals.

Example 4.11. We can write the solution to the bouncing ball problem as

$$x(t) = -\frac{1}{2}gt^2 + x_0 + \frac{I}{m}(t-T)H(t-T)$$

which looks exactly like the result we would expect from integrating the original equation twice:

$$\begin{aligned} m\ddot{x} &= -mg + I\delta(t-T) \\ m\dot{x} &= -mgt + IH(t-T) + C_1 \\ mx &= -\frac{1}{2}mgt^2 + I\cdot(t-T)H(t-T) + C_1 t + C_0 \end{aligned}$$

noting that

$$\int_{-\infty}^{x} H(t)\,\mathrm{d}t = \begin{cases} 0 & x \le 0 \\ x & x > 0 \end{cases}$$

As well as cropping up in solutions, this function is useful for posing problems with a state change, as in when a switch is thrown.

Example 4.12. Consider a simple harmonic oscillator $y(t)$ which has a sinusoidal force applied to it after $t = 2\pi$ at the resonant frequency of the system:

$$y'' + 4y = \sin(2t)H(t-2\pi)$$

Let the initial conditions be $y(0) = 1$ and $y'(0) = 0$. We can solve the problem as before by breaking it into the two natural cases:

(i) For $t < 2\pi$ we have $y'' + 4y = 0$, and using the initial conditions we have

$$y = \cos(2t)$$

(ii) For $t > 2\pi$ we have $y'' + 4y = \sin(2t)$ with general solution

$$y = A\cos(2t) + B\sin(2t) - \frac{1}{4}t\cos(2t)$$

(iii) By continuity (arguing as before that a discontinuous solution would not satisfy the equation), we have

$$\begin{aligned} \cos(4\pi) &= A\cos(4\pi) + B\sin(4\pi) - \frac{1}{4}\cdot 2\pi\cos(4\pi) \\ 1 &= A - \frac{\pi}{2} \\ A &= 1 + \frac{\pi}{2} \end{aligned}$$

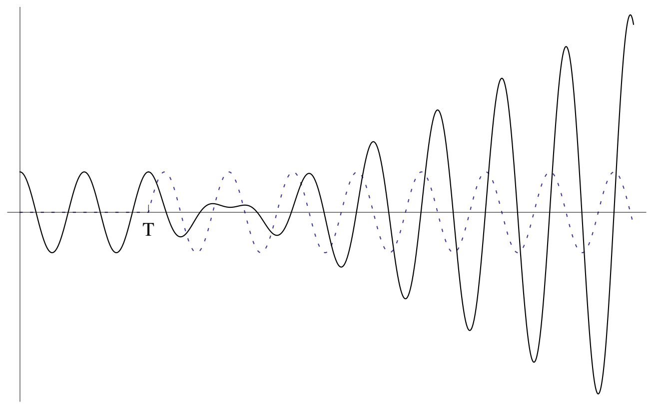


Figure 4.9: The solution to the switched resonance problem

(iv) Integrating over $[2\pi - \epsilon, 2\pi + \epsilon]$ we have

$$\Delta\left[y'\right] + 4\int_{2\pi-\epsilon}^{2\pi+\epsilon} y\mathrm{d}t = \int_{2\pi-\epsilon}^{2\pi+\epsilon} \sin\left(2t\right) H\left(t - 2\pi\right)\mathrm{d}t$$

Now the integral of y vanishes as $\epsilon \to 0$, because y is bounded near 2π (since it is continuous). Similarly, the right-hand side is also 0. Hence y' is also continuous (which we could have deduced from observing that if it was not then y'' would have introduce an unbalanced δ-type singularity into the equation) and hence

$$\begin{aligned} 0 &= 0 + 2B - \frac{1}{4}\cos\left(4\pi\right) + \frac{1}{2}\cdot 2\pi\sin\left(4\pi\right) \\ B &= \frac{1}{8} \end{aligned}$$

(v) So the solution is

$$\begin{aligned} y\left(t\right) &= \begin{cases} \cos 2t & t \le 2\pi \\ \left(1 + \frac{\pi}{2}\right)\cos 2t + \frac{1}{8}\sin 2t - \frac{t}{4}\cos 2t & t > 2\pi \end{cases} \\ &= \cos 2t + \begin{cases} 0 & t \le 2\pi \\ \left(\frac{\pi}{2} - \frac{t}{4}\right)\cos 2t + \frac{1}{8}\sin 2t & t > 2\pi \end{cases} \end{aligned}$$

as shown in Figure 4.9 - the dotted line indicates the forcing term, activated at $T = 2\pi$. Note how the phase of the solution curve changes to be quarter a period ahead of the forcing instead of quarter a period behind it.

The Heaviside step function is also useful in electrical switched problems.

4.3 Phase Space

It is useful to look in a little more detail at the language of vector spaces that we have been using, especially before we move on to solving non-linear equations.

The first thing we need to know is how to encode functions.

4.3.1 Solution vectors

A differential equation of nth order determines[7] the nth derivative $y^{(n)}(x)$ of any solution $y(x)$ in terms of $y(x)$, $y'(x)$ and all other derivatives up to $y^{(n-1)}(x)$. In fact, by differentiating n times, we can also work out all higher derivatives.

As a result, all derivatives of y at a point can be worked out using just these n values. So assuming that the solution has a globally converging Taylor expansion, we have a complete description of the function everywhere!

$$y(x) = \sum_{n=0}^{\infty} \frac{y^{(n)}(x_0)}{n!} (x - x_0)^n$$

We can think in terms of a *solution vector*

$$\mathbf{Y}(x) = \begin{pmatrix} y(x) \\ y'(x) \\ \vdots \\ y^{(n-1)}(x) \end{pmatrix}$$

defining a point for each x in an n-dimensional phase space, which is a vector space. As x varies from x_0, $\mathbf{Y}(x_0)$ contains all the information needed to trace out a trajectory in phase space.

Hence for every point at which the equations have a solution, there is a *unique* path in phase space passing through that point. It follows that two different trajectories cannot cross (because if $\mathbf{Y}_1(x_1) = \mathbf{Y}_2(x_1)$ then $\mathbf{Y}_1(x) = \mathbf{Y}_2(x)$ for all x, so the trajectories are identical).

Example 4.13. Consider for example the equation $y'' + y'/2 + 5y/16 = 0$, which has solutions $y = e^{-x/4 \pm ix/2}$. We rewrote terms like this in terms of the $\cos(\cdot)$ and $\sin(\cdot)$ basis before, as in

$$\begin{aligned} y_1 &= e^{-x/4} \cos \frac{x}{2} \\ y_2 &= e^{-x/4} \sin \frac{x}{2} \end{aligned}$$

which is useful because it means that we can restrict everything to the real case for our purposes. The solution vectors corresponding to the basis we have chosen are

$$\mathbf{Y}_1 = \begin{pmatrix} e^{-x/4} \cos \frac{x}{2} \\ -\frac{1}{4} e^{-x/4} \left(\cos \frac{x}{2} + 2 \sin \frac{x}{2} \right) \end{pmatrix}$$

[7] In general, actually, this does not hold: we are assuming the equation can be written as $y^{(n)} = f\left(t, y, y', \cdots, y^{(n-1)}\right)$ where f is some single-valued function. For example, $(y')^2 = 4y$ has two solutions passing through $(0, 1)$, corresponding to $y = (x \pm 1)^2$.

and

$$\mathbf{Y}_2 = \begin{pmatrix} e^{-x/4}\sin\frac{x}{2} \\ \frac{1}{4}e^{-x/4}\left(2\cos\frac{x}{2} - \sin\frac{x}{2}\right) \end{pmatrix}$$

These are plotted in Figure 4.10, with y on the horizontal axis and y' on the vertical axis.

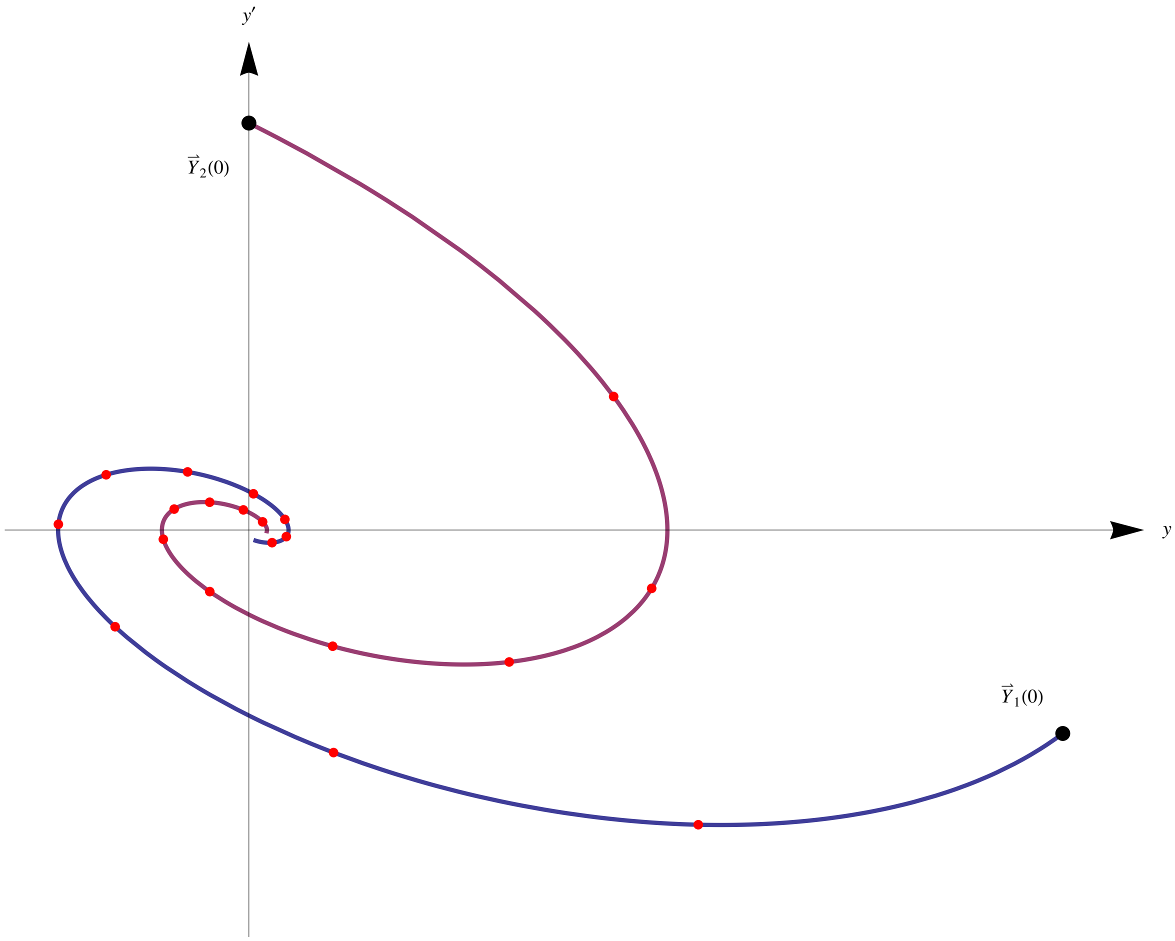


Figure 4.10: The trajectories from Example 4.13. The red dots are spaced at equal values of x.

We will use the idea of solution vectors in section 4.7 (and more generally in section 6) to transform between higher-order equations and systems of first-order equations (or equivalently, first-order equations for vectors), and we will also use them in order to reformulate many of our techniques in a concise matrix representation. For now, though, we will content ourselves with a particular observation about how these particular vectors behave, which will be useful in addressing forced equations (as we shall see in the following section on *variation of parameters*).

4.3.2 Abel's Theorem

Note that in this phase space, if the solution vectors $\mathbf{Y}_1(x)$ and $\mathbf{Y}_2(x)$ are *linearly independent at some point*, then the two solutions $y_1(x)$ and $y_2(x)$ are *independent solutions* of the differential equation.

Proof. If y_1 and y_2 are dependent, then we have $\alpha y_1 + \beta y_2 \equiv 0$ for α, β not both zero (this holds for any x under consideration). Hence $\alpha y_1' + \beta y_2' \equiv 0$ as well, so $\alpha \mathbf{Y}_1 + \beta \mathbf{Y}_2 \equiv 0$ at all points in the domain of interest.

So if $\mathbf{Y}_1$ and $\mathbf{Y}_2$ are linearly independent at some point, then the functions are linearly independent. ☐

Note the difference in the sense of 'linear independence' here: two functions are dependent if and only if one of them is a *constant multiple* of the other. There is no particular x involved here - the functions have to satisfy $y_1 \equiv \lambda y_2$ or $y_2 \equiv \mu y_1$ everywhere, so that they are trivial variations on each other. By contrast, the solution vectors have a property of linear (in)dependence for each x.

Also, note that the converse is *not* in general true: it is possible for two functions that the $\mathbf{Y}_i$ are linearly dependent everywhere, but that the functions are linearly independent. The easiest way to construct such a function is to glue together, say, x^2 and $-x^2$ at $x = 0$:

$$\begin{aligned} f_1(x) &= x^2 \\ f_2(x) &= \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases} \end{aligned}$$

Then the vectors are

$$\mathbf{Y}_1 = \begin{pmatrix} x^2 \\ 2x \end{pmatrix} \quad \mathbf{Y}_2 = \begin{pmatrix} \pm x^2 \\ \pm 2x \end{pmatrix}$$

and clearly $\mathbf{Y}_1 = \pm \mathbf{Y}_2$ depending on the sign of x, but $f_1 \not\equiv \lambda f_2$ for all x, for any value of λ, etc. - again, the different senses of linear dependence are at play here.

Remark. This result obviously extends to higher dimensions.

Now recall that a set of vectors $\mathbf{v}_1, \cdots, \mathbf{v}_k$ are linearly independent if and only if the matrix determinant $|\mathbf{v}_1 \quad \mathbf{v}_2 \quad \cdots \quad \mathbf{v}_k|$ containing the columns $\mathbf{v}_i$ is non-zero.

Definition 4.14. The *Wronskian* for a set of functions $y_i(x)$ is the determinant $W(x)$ of the matrix with the solution vectors as columns.

For the second-order case,

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

So if $W(x_0) \neq 0$ for some x_0 the solutions are independent.

Example 4.15. Using the example above, we have

$$\begin{aligned} W(x) &= \begin{vmatrix} e^{-x/4}\cos\frac{x}{2} & e^{-x/4}\sin\frac{x}{2} \\ -\frac{1}{4}e^{-x/4}\left(\cos\frac{x}{2}+2\sin\frac{x}{2}\right) & \frac{1}{4}e^{-x/4}\left(2\cos\frac{x}{2}-\sin\frac{x}{2}\right) \end{vmatrix} \\ &= \frac{1}{4}e^{-2x/4}\left(2\cos^2\frac{x}{2}-\sin\frac{x}{2}\cos\frac{x}{2}+\sin\frac{x}{2}\cos\frac{x}{2}+2\sin^2\frac{x}{2}\right) \\ &= \frac{1}{4}e^{-x/2}\cdot 2 \\ &= \frac{1}{2}e^{-x/2}\neq 0 \end{aligned}$$

Note that in this example, the Wronskian is actually non-zero *everywhere*. This, in fact, is a general rule for solutions to *linear* differential equations with a continuity conditions on the coefficients:

Theorem 4.16 (Abel's Theorem)**.** *If $W(x_0)\neq 0$ for some particular value x_0, then $W(x)\neq 0$ for any value of x, assuming that the coefficient of $y^{(n-1)}$ is a continuous function when the function is written in standard form.*

We will present here a proof of the two-dimensional case, since it is of special interest to us here. See section 6.4 for the general proof.

Proof. Write the equation as

$$y''+p(x)\,y'+q(x)\,y=0$$

so that p is continuous. This holds for y_1 and y_2.

Now consider multiplying each equation by the other solution and subtracting:

$$\begin{aligned} y_2\left(y_1''+py_1'+qy_1\right) &= 0 \\ y_1\left(y_2''+py_2'+qy_2\right) &= 0 \\ \left[y_2y_1''-y_1y_2''\right]+p\left[y_2y_1'-y_1y_2'\right] &= 0 \end{aligned}$$

But now note that the term multiplying p is precisely the negative of the Wronskian

$$-W=-\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=-y_1y_2'+y_2y_1'$$

and that also

$$\begin{aligned} W' &= y_1y_2''-y_2y_1''+y_1'y_2'-y_1'y_2' \\ &= -\left[y_2y_1''-y_1y_2''\right] \end{aligned}$$

Hence

$$\begin{aligned} -W' - pW &= 0 \\ W' + p(x)W &= 0 \\ W &= W_0 e^{-\int p(x)\mathrm{d}x} \end{aligned}$$

Then since $e^z \neq 0$ for any arbitrary complex number $z \in \mathbb{C}$, either $W_0 = 0$ and $W(x) \equiv 0$ *or* $W_0 \neq 0$ and $W(x) \neq 0$ for any x. □

Remark. We will use some of the intermediate stages in this proof again. Note that the expression

$$\boxed{W = W_0 e^{-\int p \mathrm{d}x}}$$

is sometimes known as *Abel's Identity*. (Again, in section 6.4, the generalization of this statement is given.)

4.4 Variation of Parameters

In this section we will see how to obtain particular solutions to forced non-linear equations using two complementary functions. Section 6.5 has the generalized version of this method, for higher dimensional cases.

We begin by taking a general second-order linear ODE,

$$y'' + p(x)y' + q(x)y = f(x)$$

and two linearly independent solutions $y_1(x)$ and $y_2(x)$.

Then the solution vectors

$$\mathbf{Y}_1 = \begin{pmatrix} y_1 \\ y_1' \end{pmatrix} \quad \text{and} \quad \mathbf{Y}_2 = \begin{pmatrix} y_2 \\ y_2' \end{pmatrix}$$

form a *basis* of the space of functions satisfying such a second-order equation, for every x, because we know they remain at all times linearly independent (from Theorem 4.16).

So at any point x, any solution vector $\mathbf{Y}_p$ can be written as a linear combination of these vectors:

$$\mathbf{Y}_p(x) = u(x)\mathbf{Y}_1(x) + v(x)\mathbf{Y}_2(x)$$

Note that the coordinates $u(x)$ and $v(x)$ are both functions of x (in general) - what we want now is an equation for u and v - in fact, we expect them to be determined up to at most an additive constant. But we have lots of information about the functions involved. Writing the two scalar equations implied by this vector equation out, we have

$$\begin{aligned} y_p &= uy_1 + vy_2 \\ y_p' &= uy_1' + vy_2' \end{aligned}$$

But since we know y_p is a solution to the (forced) ODE, we know

$$\begin{aligned} y_p'' + py_p' + qy_p &= f \\ (uy_1' + vy_2')' + p(uy_1' + vy_2') + q(uy_1 + vy_2) &= f \end{aligned}$$

But y_1 and y_2 are solutions to the homogeneous equation, so we can rewrite this as

$$\begin{aligned} f &= u(y_1'' + py_1' + qy_1) + v(y_2'' + py_2' + qy_2) + y_1'u' + y_2'v' \\ &= y_1'u' + y_2'v' \end{aligned}$$

This is the first equation we will use to compute u' and v'. To obtain another, we can use the original two equations:

$$\begin{aligned} uy_1' + vy_2' &= y_p' \\ &= (uy_1 + vy_2)' \\ &= u'y_1 + uy_1' + v'y_2 + vy_2' \\ 0 &= u'y_1 + v'y_2 \end{aligned}$$

So we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

which we can solve by simply inverting the matrix on the left (its determinant is $W \neq 0$) to get

$$\begin{aligned} \begin{pmatrix} u' \\ v' \end{pmatrix} &= \frac{1}{W} \begin{pmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{pmatrix} \begin{pmatrix} 0 \\ f \end{pmatrix} \\ u' &= -\frac{y_2}{W} f \\ v' &= \frac{y_1}{W} f \end{aligned}$$

We can then integrate these equations in order to obtain the u and v required.

The full result can be written in the form

$$\boxed{y_p = y_1 \int -\frac{y_2 f}{W} \mathrm{d}x + y_2 \int \frac{y_1 f}{W} \mathrm{d}x}$$

noting that the additive constants in the integrals correspond to adding on multiples of complementary functions, which was as we expected.

Remark. This method can be generalized to equations of higher order, and we do exactly that in section 6.5. The key idea is to write the particular solution as a combination of the homogeneous ones.

$$\boxed{\mathbf{Y}_p(x) = u(x)\mathbf{Y}_1(x) + v(x)\mathbf{Y}_2(x)}$$

Example 4.17. Solve $\ddot{y} + \omega^2 y = \sin \omega t$.

We will demonstrate the full technique here for completeness.

We have two solutions, $y_1 = \sin \omega t$ and $y_2 = \cos \omega t$. The Wronskian for these solutions is

$$\begin{vmatrix} \sin \omega t & \cos \omega t \\ \omega \cos \omega t & -\omega \sin \omega t \end{vmatrix} = -\omega$$

Choose u and v so that $y_p = u \sin \omega t + v \cos \omega t$ and $\dot{y}_p = u\omega \cos \omega t - v\omega \sin \omega t$. But then

$$\begin{aligned} \dot{y}_p &= u\omega \cos \omega t + \dot{u} \sin \omega t \\ &\quad -v\omega \sin \omega t + \dot{v} \cos \omega t \end{aligned}$$

so

$$(\sin \omega t)\, \dot{u} + (\cos \omega t)\, \dot{v} = 0$$

Also, differentiating y_p' we have

$$\begin{aligned} \ddot{y}_p &= -u\omega^2 \sin \omega t + \dot{u}\omega \cos \omega t \\ &\quad -v\omega^2 \cos \omega t - \dot{v}\omega \sin \omega t \end{aligned}$$

and substituting this into the original equation we have

$$(\omega \cos \omega t)\, \dot{u} - (\omega \sin \omega t)\, \dot{v} = \sin \omega t$$

Combining these two equations, we have

$$\begin{aligned} \dot{u} &= \frac{\cos \omega t \sin \omega t}{\omega} = \frac{\sin 2\omega t}{2\omega} \\ \dot{v} &= -\frac{\sin^2 \omega t}{\omega} = \frac{\cos 2\omega t - 1}{2\omega} \end{aligned}$$

Hence, integrating and ignoring any constant factors, we see

$$\begin{aligned} u &= -\frac{\cos 2\omega t}{4\omega^2} \\ v &= \frac{\sin 2\omega t}{4\omega^2} - \frac{t}{2\omega} \end{aligned}$$

Thus the solution we have found is

$$\begin{aligned} y_p &= \frac{1}{4\omega^2}\left(-\cos 2\omega t \sin \omega t + \sin 2\omega t \cos \omega t\right) - \frac{t}{2\omega} \cos \omega t \\ &= \underbrace{\frac{\sin \omega t}{4\omega^2}}_{\text{multiple of complementary function}} - \underbrace{\frac{t}{2\omega} \cos \omega t}_{\text{as found by detuning}} \end{aligned}$$

which is, up to the arbitrary multiples of the complementary function, precisely the result we found from detuning in section 4.2.1.

Recall also that in section 4.2.1 that we tried to solve

$$y'' - 2y' + y = e^x + xe^x$$

and discovered that x^2e^x and x^3e^x gave us the necessary remainders. We can deduce these general rules for second-order equations using variation of parameters.

Theorem 4.18.

(i) *If $e^{\lambda_1 x}$ and $e^{\lambda_2 x}$ are two independent complementary functions for a second-order linear equation, then for an equation with forcing term $g(x)$ there is a particular integral of the form*

$$\left(-e^{\lambda_1 x}\left[\int^x e^{-\lambda_1 t} g(t)\,\mathrm{d}t\right] + e^{\lambda_2 x}\left[\int^x e^{-\lambda_2 t} g(t)\,\mathrm{d}t\right]\right) / (\lambda_2 - \lambda_1)$$

(ii) *If $e^{\lambda x}$ and $xe^{\lambda x}$ are two independent complementary functions for a second-order linear equation, then for an equation with forcing term $g(x)$ there is a particular integral of the form*

$$e^{\lambda x}\left(-\int^x te^{-\lambda t} g(t)\,\mathrm{d}t + x\int^x e^{-\lambda t} g(t)\,\mathrm{d}t\right)$$

Proof.

(i) We have the Wronskian

$$\begin{aligned} W &= \begin{vmatrix} e^{\lambda_1 x} & e^{\lambda_2 x} \\ \lambda_1 e^{\lambda_1 x} & \lambda_2 e^{\lambda_2 x} \end{vmatrix} \\ &= e^{\lambda_1 x} e^{\lambda_2 x} (\lambda_2 - \lambda_1) \end{aligned}$$

so using the results from above,

$$\begin{aligned} y_p &= y_1 \int -\frac{y_2 f}{W}\mathrm{d}x + y_2 \int \frac{y_1 f}{W}\mathrm{d}x \\ &= -e^{\lambda_1 x}\int \frac{e^{\lambda_2 x} g(x)}{e^{\lambda_1 x} e^{\lambda_2 x} (\lambda_2 - \lambda_1)}\mathrm{d}x + e^{\lambda_2 x}\int \frac{e^{\lambda_1 x} g(x)}{e^{\lambda_1 x} e^{\lambda_2 x} (\lambda_2 - \lambda_1)}\mathrm{d}x \\ &= -e^{\lambda_1 x}\int \frac{e^{-\lambda_1 x} g(x)}{(\lambda_2 - \lambda_1)}\mathrm{d}x + e^{\lambda_2 x}\int \frac{e^{-\lambda_2 x} g(x)}{(\lambda_2 - \lambda_1)}\mathrm{d}x \end{aligned}$$

as required.

(ii) The Wronskian is now

$$\begin{aligned} W &= \begin{vmatrix} e^{\lambda x} & xe^{\lambda x} \\ \lambda e^{\lambda x} & (1+\lambda x)\, e^{\lambda x} \end{vmatrix} \\ &= e^{2\lambda x}(1+\lambda x-\lambda x) \\ &= e^{2\lambda x} \end{aligned}$$

so

$$\begin{aligned} y_p &= y_1 \int -\frac{y_2 f}{W}\mathrm{d}x + y_2 \int \frac{y_1 f}{W}\mathrm{d}x \\ &= -e^{\lambda x} \int \frac{xe^{\lambda x} g(x)}{e^{2\lambda x}}\mathrm{d}x + xe^{\lambda x} \int \frac{e^{\lambda x} g(x)}{e^{2\lambda x}}\mathrm{d}x \\ &= -e^{\lambda x} \int xe^{-\lambda x} g(x)\,\mathrm{d}x + xe^{\lambda x} \int e^{-\lambda x} g(x)\,\mathrm{d}x \end{aligned}$$

□

These general formulae mean that we can write down solutions for *any* (continuous, at least) forcing term in a linear second-order equation with constant coefficients in the form of an integral.

Remark. In particular, this gives us a particular solution for a forcing term proportional to $x^k e^{\lambda x}$ for an equation with a repeated root:

$$\begin{aligned} y_p &= e^{\lambda x}\left[-\int x \cdot x^k \mathrm{d}x + x \int x^k \mathrm{d}x\right] \\ &= a e^{\lambda x} x^{k+2} \end{aligned}$$

where we ignore multiples of the complementary functions.

Note that for the case of distinct roots, we can get the general results for $x^k e^{\lambda_1 x}$ too. Ignoring constant factors,

$$\begin{aligned} y_p &= -e^{\lambda_1 x}\left[\int e^{-\lambda_1 x} x^k e^{\lambda_1 x} \mathrm{d}x\right] + e^{\lambda_2 x}\left[\int e^{-\lambda_2 x} x^k e^{\lambda_1 x}\mathrm{d}x\right] \\ &= -e^{\lambda_1 x}\left[a x^{k+1}\right] + e^{\lambda_2 x}\left[\int e^{(\lambda_1-\lambda_2)x} x^k \mathrm{d}x\right] \end{aligned}$$

Now if $k > 0$ is an integer, this last integral evaluates (as may be shown by integration by parts) to a kth degree polynomial in x multiplied by the same exponential, and the overall expression is a $(k+1)$th degree polynomial in x multiplied by $e^{\lambda_1 x}$. Otherwise, the expression can only be written in terms of the (incomplete) gamma function

$$\Gamma(a; z) = \int_z^\infty t^{a-1} e^{-t} \mathrm{d}t$$

4.5 Equidimensional Equations

An *equidimensional*[8] equation is an ODE which is invariant under the transformation $x \to \mu x$ for any $\mu \neq 0$ - often, we also require that the equation is linear, and sometimes that it is not forced. In this section, we will consider linear equidimensional functions which are (in general) forced, and by terms which do *not* have to respect the scale invariance.

In this case, it is equivalent to the requirement derivatives of the nth order always appear multiplied by x^n, since only then do we get

$$x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} \to (\mu x)^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n} \cdot \frac{1}{\mu^n} = x^n \frac{\mathrm{d}^n y}{\mathrm{d}x^n}$$

We can write an essentially general (up to, for example, $x \to x + \delta$) second-order equation as

$$\boxed{ax^2 y'' + bxy' + cy = f(x)}$$

for a, b, c all constant.

4.5.1 Solving the equation

This simple form of equation admits a general solution - we are essentially interested in eigenfunctions of the new operator $x\frac{\mathrm{d}}{\mathrm{d}x}$, and its square,

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left[x\frac{\mathrm{d}}{\mathrm{d}x}\right] \equiv x\left[x\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \frac{\mathrm{d}}{\mathrm{d}x}\right] = x^2\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x\frac{\mathrm{d}}{\mathrm{d}x}$$

But we can see directly from the original operator that any polynomial term x^k is an eigenfunction, because the derivative reduces the power of x by one, and the multiplication restores it:

$$x\frac{\mathrm{d}}{\mathrm{d}x}\left[x^k\right] = kx \cdot x^{k-1} = kx^k$$

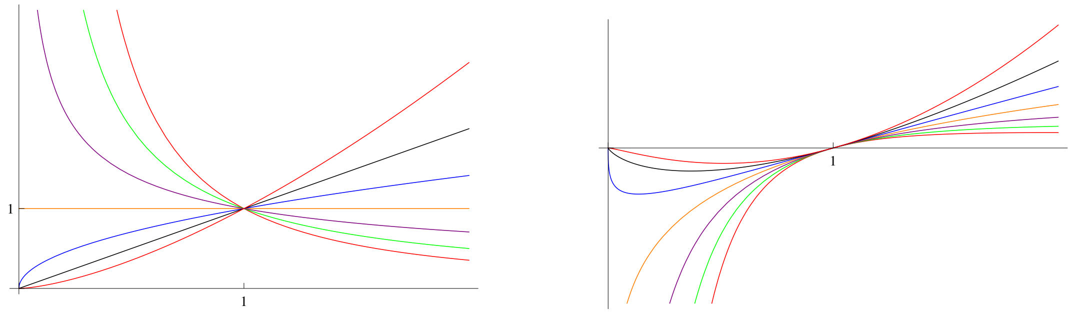
So to solve the unforced version of the original equation, we can guess $y = x^k$. We know that we will then have

$$\begin{aligned} a\left[x^2\frac{\mathrm{d}^2\left(x^k\right)}{\mathrm{d}x^2} + x\frac{\mathrm{d}\left(x^k\right)}{\mathrm{d}x}\right] + bx\frac{\mathrm{d}\left(x^k\right)}{\mathrm{d}x} + c\left(x^k\right) &= 0 \\ ak(k-1) + bk + c &= 0 \\ ak^2 + (b-a)k + c &= 0 \end{aligned}$$

This is the characteristic equation for the eigenvalue k, which may be solved to find $k = k_1, k_2$, and then *if* k_1 *and* k_2 *are distinct*, we can write

$$y_c = Ax^{k_1} + bx^{k_2}$$

[8] Confusingly, equidimensional equidimensional equations are also called "homogeneous" equations, which has a different meaning here to "unforced". Other names include a *Cauchy-Euler*, *Euler-Cauchy* and *Euler's* equation.



(a) Distinct roots (not multiplied by $\ln x$) (b) Repeated roots (multiplied by $\ln x$)

Figure 4.11: Example solutions to the logistic equation for real roots

However, this is no help at all if the two roots are the same. Clearly, multiplying the function by x will not work in this case!

Instead, we must use what we have learned from finding the typical complementary functions to work out how to solve the equation. The main idea is to note that the basis functions x^k have the form of $e^{k \ln x}$, so perhaps if we rewrote the equation in terms of $z = \ln x$ we would find an equation that we already know how to solve.

The transformation maps $x \to e^z$, and $\frac{\mathrm{d}}{\mathrm{d}x} \to \frac{\mathrm{d}}{\mathrm{d}e^z} = \frac{\mathrm{d}}{\mathrm{d}z} / \frac{\mathrm{d}e^z}{\mathrm{d}z} = e^{-z}\frac{\mathrm{d}}{\mathrm{d}z}$ so the new equation has the form

$$\begin{aligned} ae^{2z}e^{-z}\frac{\mathrm{d}}{\mathrm{d}z}\left[e^{-z}\frac{\mathrm{d}y}{\mathrm{d}z}\right] + be^z e^{-z}\frac{\mathrm{d}y}{\mathrm{d}z} + cy &= f\left(e^z\right) \\ ae^z\left[e^{-z}\frac{\mathrm{d}^2y}{\mathrm{d}z^2} - e^{-z}\frac{\mathrm{d}y}{\mathrm{d}z}\right] + b\frac{\mathrm{d}y}{\mathrm{d}z} + cy &= f\left(e^z\right) \\ a\frac{\mathrm{d}^2y}{\mathrm{d}z^2} + (b-a)\frac{\mathrm{d}y}{\mathrm{d}z} + cy &= f\left(e^z\right) \end{aligned}$$

The left-hand side now has precisely the form of a second-order linear equation with constant coefficients. We can see that this has the same characteristic equation as the one we found for k above, with complementary functions $e^{kz} \equiv x^k$, and what is more, we know that in the case that the roots coincide, the new solution is

$$ze^{kz} \equiv (\ln x)\, x^k$$

The overall shape of both forms of solution for real k are shown in Figure 4.11.

As a point of interest, note the form of solutions for complex roots k:

$$\begin{aligned} y &= e^{(\alpha+\beta i)z} \\ &= e^{\alpha z}\left[\cos\beta z + \sin\beta z\right] \\ &= x^\alpha\left[\cos\left(\beta\ln x\right) + i\sin\left(\beta\ln x\right)\right] \end{aligned}$$

Hence we can take $x^\alpha\cos\left(\beta\ln x\right)$ and $x^\alpha\sin\left(\beta\ln x\right)$ to be our two independent solutions. Typical solutions of the first form in this case are shown in Figure 4.12.

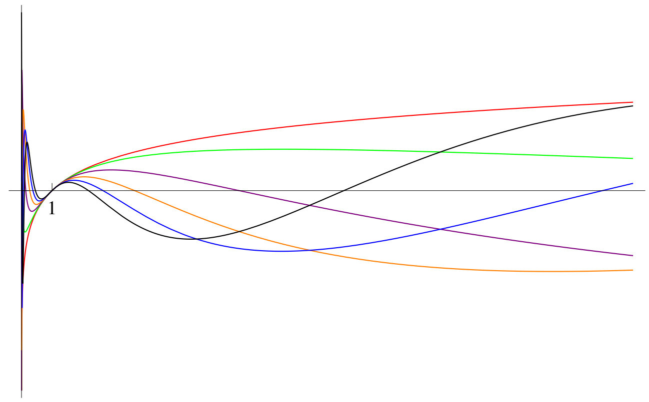


Figure 4.12: Equidimensional solution curves for complex roots

Remark. Forced versions of this equation are probably best solved by applying techniques we have already discussed to the transformed version of the equation.

4.5.2 Difference equation analogues

There are two ways in which we can view the discrete version of an equidimensional equation.

Typical case Firstly, consider a general linear recurrence relation of the form

$$ay_{n+2} + by_{n+1} + cy_n = f_n$$

We can solve this by finding the form of eigenfunctions of the map $y_n \to y_{n+1}$, and exploiting the linearity of the equation.

But clearly, the eigenfunctions of $y_n \to y_{n+1}$ are just $y_n = k^n$, with the corresponding eigenvalue k, because then

$$y_{n+1} = k \cdot k^n = ky_n$$

Remark. Alternatively, the *(forward) difference operator* $D\left[y_n\right]$ is defined by

$$D\left[y_n\right] = y_{n+1} - y_n$$

and it too has eigenfunctions $y_n = k^n$:

$$\begin{aligned} D\left[y_n\right] &= k^{n+1} - k^n \\ &= (k-1)\, y_n \end{aligned}$$

So to solve the original equation, we can think of it as

$$AD^2\left[y_n\right] + BD\left[y_n\right] + Cy_n = f_n$$

where $A = a$, $B = b + 2a$ and $C = c + b + a$, and then solve from there.

Either way, we will end up with the clear characteristic equation

$$\begin{aligned} ak^{n+2} + bk^{n+1} + ck^n &= 0 \\ ak^2 + bk + c &= 0 \end{aligned}$$

with solutions $k = k_1, k_2$.

The general complementary functions are then

$$y_n = Ak_1^n + Bk_2^n$$

for distinct roots.

In the case of a repeated root, it is not very surprising that we get solutions of the form

$$y_n = (A + Bn)\, k^n$$

Particular integrals are of broadly the same form as we have discussed previously:

Inhomogeneity f_n	Term of particular solution y_n
k^n for root k	$k^n, n \cdot k^n, \cdots$
b^n	b^n
n^p $an^p + bn^{p-1} + \cdots$	$An^p + Bn^{p-1} + \cdots + Cn + D$

Example 4.19. Consider the modified Fibonacci sequence $G_n = G_{n-1} + G_{n-2} + 1$, with initial conditions $G_0 = 0$ and $G_1 = 1$. This has terms

$$0, 1, 2, 4, 7, 12, 20, 33, \cdots$$

The characteristic equation is $k^2 - k - 1 = 0$ which has solutions

$$\begin{aligned} k &= \frac{1 \pm \sqrt{5}}{2} \\ &= \Phi, -\frac{1}{\Phi} \end{aligned}$$

Let us first try to find a particular solution. Referring to the above table, we see the forcing is of the form 1^n, so we guess a particular solution

$$G_n = a \cdot 1^n = a$$

which is easily found to be correct for

$$\begin{aligned} a &= a + a + 1 \\ 0 &= a + 1 \\ a &= -1 \end{aligned}$$

Then we have

$$G_n = A\Phi^n + B\left(-\frac{1}{\Phi}\right)^n - 1$$

Now we can find A and B:

$$\begin{aligned} G_0 &= A + B - 1 = 0 \\ G_1 &= \frac{(1+\sqrt{5})}{2}A + \frac{(1-\sqrt{5})}{2}B - 1 \\ &= \frac{A+B}{2} + \left(\frac{A-B}{2}\right)\sqrt{5} - 1 \\ &= 1 \end{aligned}$$

So we have

$$\begin{aligned} A + B &= 1 \\ A - B &= \frac{3}{\sqrt{5}} \\ A &= \frac{3+\sqrt{5}}{2\sqrt{5}} = \frac{\Phi^2}{\sqrt{5}} \\ B &= \frac{3-\sqrt{5}}{2\sqrt{5}} = \frac{(-1/\Phi)^2}{\sqrt{5}} \end{aligned}$$

which gives

$$\begin{aligned} G_n &= \frac{\Phi^{n+2} + \left(-\frac{1}{\Phi}\right)^{n+2}}{\sqrt{5}} - 1 \\ &= F_{n+2} - 1 \end{aligned}$$

Indeed, if we add 1 to the sequence for G_n we get

$$1, 2, 3, 5, 8, 13, 21, 34, \cdots$$

which is precisely the Fibonacci sequence shifted by two places (assuming $F_0 = 0$ and $F_1 = 1$).

Alternative analogy The other way in which it is possible to draw an analogy with a discrete equation is via the more complicated expression given, for a fixed integer m, by

$$f_m(n) = n(n+1)\cdots(n+m-1)$$

The point about this function is the properties it has with respect to the forward difference operator $D[\cdot]$:

$$\begin{aligned} D[f_m(n)] &= f_m(n+1) - f_m(n) \\ &= (n+1)(n+2)\cdots(n+m-1)[(n+m)-n] \\ &= \frac{m}{n} f_m(n) \end{aligned}$$

It can be shown that repeating this process k times gives us

$$\begin{aligned} D^k[f_m(n)] = f_m^{(k)}(n) &= \frac{m(m-1)\cdots(m-k+1)}{n(n+1)\cdots(n+k-1)} f_m(n) \\ &= m(m-1)\cdots(m-k+1)\frac{f_m(n)}{f_k(n)} \end{aligned}$$

This is very similar to the fact that

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}[x^m] = m(m-1)\cdots(m-k+1)\frac{x^m}{x^k}$$

which is what we used to solve the equidimensional equation, in which terms of the form $x^k \frac{\mathrm{d}^k y}{\mathrm{d}x^k}$ appear. In fact, we can use this to solve equations like

$$an(n+1)y_n^{(2)} + bny_n^{(1)} + cy_n = 0$$

because we can guess solutions like $y_n = f_m(n)$ and solve for m:

$$\begin{aligned} af_m(n)\cdot m(m-1) + bf_m(n)\cdot m + cf_m(n) &= 0 \\ am(m-1) + bm + c &= 0 \end{aligned}$$

which is the same form of the characteristic equation as we had before.

Example 4.20. Solve $n(n+1)y_{n+2} - 2n(n+1)y_{n+1} - (2+n(n+1))y_n = 0$.

This is not currently of the stated form, since we need terms like $y_n^{(2)}$ and so on. Let us rewrite this:

$$n(n+1)\left[y_n^{(2)} + 2y_{n+1} - y_n\right] - 2n(1+n)y_{n+1} - (2+n(n+1))y_n = n(n+1)y_n^{(2)} - 2y_n = 0$$

Now it is as required, so we can attempt a solution $y_n = f_m(n)$. We need to solve

$$\begin{aligned} m(m-1)-2 &= 0 \\ (m+1)(m-2) &= 0 \\ m &= -1, 2 \end{aligned}$$

Hence our general solution is

$$y_n = \frac{A}{n-1} + Bn(n+1)$$

demonstrating that this works for $m < 0$.

Remark. This approach also works for non-integer m if we use the Γ function instead of the partial factorials:

$$f_m(n) = \frac{\Gamma(n+m)}{\Gamma(n)}$$

4.6 Series Solutions

So far, we have mainly looked at special cases of linear second-order equations, notably those with constant coefficients, and equidimensional equations. However, it is obviously not always the case that we can find an analytic solution.

In section 3.6, we developed chiefly graphical methods for analyzing first-order equations. Here, we will develop a more algebraic approach to solving equations like

$$a(x)y'' + b(x)y' + c(x)y = 0$$

or, where appropriate,

$$y'' + p(x)y' + q(x)y = 0$$

The idea is that we try to find solutions as (in general infinite) series in terms of powers of the independent variable x. We are most familiar with one type of power series, Taylor series, where the powers are all non-negative integers, as in

$$\sum_{n=0}^{\infty} a_n (x - x_0)^n$$

though for the rest of this section we will work mainly with the translated case $x_0 = 0$, so that this would be the Maclaurin series

$$\sum_{n=0}^{\infty} a_n x^n$$

It turns out that it is not always possible to solve equations in this form, but we can often find a similar form of solution. It will be useful to classify points according to the behaviour of the series for the coefficients as follows:

Definition 4.21. $x = x_0$ is an *ordinary point* if $p(x) \equiv \frac{b(x)}{a(x)}$ and $q(x) \equiv \frac{c(x)}{a(x)}$ have Taylor series about x_0.

Otherwise, x_0 is a *singular point*. If x_0 is singular, but the equation may be written in the form

$$A(x)(x-x_0)^2 y'' + B(x)(x-x_0) y' + C(x) y = 0$$

where $\frac{B(x)}{A(x)} \equiv (x-x_0) p(x)$ and $\frac{C(x)}{A(x)} = (x-x_0)^2 q(x)$ have Taylor series here, then we say x_0 is a *regular singular point* - otherwise, it is *irregular*.

Remark. Another way the difference between the types of singular points is commonly expressed is that a regular singular point has p and q with *poles* at most order 1 and 2 respectively. Note that we rewrite the equation in equidimensional form to see its behaviour.

Recall that the Taylor series expansion must exist and converge for all x in some interval containing x_0, but not necessarily everywhere.

Here are a few examples for clarity:

Example 4.22. You can check the following cases as an exercise.

(i) $(1-x^2) y'' - xy' + 5y = 0$. Here, $x = 0$ is ordinary, and $x = \pm 1$ are regular singular points.

(ii) $(\sin x) y'' + (\cos x) y' + 5y = 0$. All points of the form $x = n\pi$ for integer n are singular, but they are in fact regular.

(iii) $(1+\sqrt{x}) y'' - xy' + 5y = 0$. The point $x = 0$ is now an example of an *irregular* singular point.

4.6.1 Taylor series solutions

Let's first look at an example of an equation which can be solved using a Taylor series.

Example 4.23. Solve

$$y'' + xy' - 2y = 0$$

Let us expand about the point $x = 0$ (for no reason other than it makes the bookkeeping easier), so that our trial (Taylor series) solution is

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then the equation can be written in terms of an infinite sum, since for example

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}$$

However, to make the manipulations clearer, we will consider the equation in the pseudo-equidimensional form we used in classifying stationary points:

$$\left(x^2 y''\right) + x^2\left(xy'\right) - 2x^2\left(y\right) = 0$$

Then we have

$$\sum_{n=0}^{\infty}\left[n\left(n-1\right) + nx^2 - 2x^2\right] a_n x^n = 0$$

We want to extract some information about the coefficients a_n, but this is obviously difficult since the powers of x are interleaved arbitrarily into the equation. To try to solve this problem, let us reorganize the terms into the following form:

$$\sum_{n=0}^{\infty}\left[n\left(n-1\right) a_n + a_{n-2}\left(n-2\right) - 2a_{n-2}\right] x^n = 0$$

where we have just realized that terms like $f\left(n\right) x^2 \cdot x^n = f\left(n\right) x^{n+2}$ can equally well be written as $f\left(n-2\right) x^n$, just appearing 2 places later in the sequence. We have invented new coefficients a_{-1} and a_{-2}, which are both 0, in order to enable us to do this without separating out the $n = 0, 1$ cases.

But now since both sides are identical, we can just compare coefficients of x^n!

$$\begin{aligned} n\left(n-1\right) a_n + a_{n-2}\left(n-2\right) - 2a_{n-2} &= 0 \\ n\left(n-1\right) a_n &= \left(4-n\right) a_{n-2} \end{aligned}$$

This allows us to work out the coefficients in the power series, provided we have some initial terms, a_0 and a_1, as for $n \neq 0, 1$ we have

$$a_n = \frac{4-n}{n\left(n-1\right)} a_{n-2}$$

so

$$\begin{aligned} a_2 &= \frac{2}{2} a_0 \\ a_4 &= \frac{0}{4 \cdot 3} a_2 = 0 \\ a_{2m} &= 0 \end{aligned}$$

and

$$\begin{aligned} a_3 &= \frac{1}{3 \cdot 2} a_1 \\ a_5 &= \frac{-1}{5 \cdot 4} a_3 = -\frac{1}{5 \cdot 4 \cdot 3 \cdot 2} a_1 \\ a_7 &= \frac{-3}{7 \cdot 6} a_5 = \frac{3}{7!} a_1 \\ a_{2m+1} &= \left(-1\right)^{m+1} \frac{1 \cdot 3 \cdot \cdots \cdot \left(2m-3\right)}{\left(2m+1\right)!} a_1 \quad \text{for } m > 1 \end{aligned}$$

Hence if we take the case $a_0 = 1$, $a_1 = 0$ we get the solution

$$y_1 = 1 + x^2$$

and if we take $a_0 = 0$, $a_1 = 1$ we get

$$\begin{aligned} y_2 &= x + \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{1680}x^7 - \cdots \\ &\cdots + (-1)^{m+1} \frac{1 \cdot 3 \cdot \cdots \cdot (2m-3)}{(2m+1)!} x^{2m+1} + \cdots \end{aligned}$$

These two solutions can then be used as the basis of a general solution

$$y = Ay_1 + By_2$$

exactly as before.

This gives the reader a fairly good example of what to expect in many situations - sometimes one or more solutions will have a polynomial expression; sometimes they will be infinite, in which case the solutions will converge over at least the same range as the coefficients $p(x)$ and $q(x)$ do (see next section) - hence in the case of functions like these, which are everywhere equal to their Taylor series about any point, the solutions should also be universally defined by a single Taylor series. In general, *analytic* coefficients which are *locally* equal to Taylor series about any point have analytic solutions. Infinite expressions can also be sometimes be identified in closed form, though in general we have no reason to expect they can - there is no natural way to express the second solution above.

It is a general result (due to Frobenius and Fuchs) that if x_0 is a regular point, there will be Taylor series solutions:

Theorem 4.24. *If $p(x)$ and $q(x)$ both have Taylor series in some interval about x_0, then there are two independent solutions to the equation $y'' + p(x)y' + q(x)y = 0$ of the form*

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

which converge over at least the same region as p and q.

We will not prove this result here.

For an example where p and q do not have Taylor series that converge everywhere, consider the following example:

Example 4.25. Solve

$$(1-x)^2 y'' - (1-x)y' - y = 0$$

We will expand around $x = 0$ again, noting that $p = 1/(1-x)$ and $q = 1/(1-x)^2$ both have

Taylor expansions at this point which are only valid for $|x| < 1$. Again, assume there is a locally convergent solution

$$y = \sum a_n x^n$$

and proceed to adjust the equation and compare coefficients:

$$\left(1 - x^2\right)\left(x^2 y''\right) - \left(x - x^2\right)\left(xy'\right) - x^2 y = 0$$

Thus:

$$\begin{aligned}
\sum \left[n(n-1) - 2n(n-1)x + n(n-1)x^2 - nx + nx^2 - x^2\right] a_n x^n &= 0 \\
\sum \left[n(n-1) - n(2n-1)x + (n-1)(n+1)x^2\right] a_n x^n &= 0 \\
\sum \left[n(n-1)a_n - (n-1)(2n-3)a_{n-1} + (n-3)(n-1)a_{n-2}\right] x^n &= 0 \\
(n-1)\left[na_n - (2n-3)a_{n-1} + (n-3)a_{n-2}\right] &= 0
\end{aligned}$$

This equation looks unpleasant, but it is actually quite easy to pick out simple forms for the solution by clever choices of a_0 and a_1.

First, note that $2a_2 - a_1 - a_0 = 0$, and that $3a_3 - 3a_2 = 0$. The second equation tells us $a_3 = a_2$. Note that if $a_{n-1} = a_{n-2}$ then we have

$$\begin{aligned}
na_n + (n - 3 - 2n + 3)a_{n-1} &= 0 \\
na_n &= na_{n-1} \\
a_n &= a_{n-1}
\end{aligned}$$

Hence all series solutions have constant terms beyond a_2! So the natural first choice is to make these all vanish, which from the first equation happens when $a_1 + a_0 = 0$. Indeed, $a_0 = 1$ and $a_1 = -1$ gives us the solution

$$y = 1 - x$$

For the other solution, let us choose the neat solution $a_0 = 1$, $a_1 = 1$ since then all terms are identically 1:

$$y = 1 + x + x^2 + x^3 + \cdots$$

This solution we can recognize instantly as being the Taylor series, valid for $|x| < 1$, for

$$y = \frac{1}{1 - x}$$

It is easy to verify that both of these closed-form solutions work for all x, even outside the radius of convergence of the coefficients, though the power series is only valid for the same range as those, namely $|x| < 1$.

Remark. We could have solved the above equation analytically using the methods from the section on equidimensional equations: making the translation $u = x - 1$ we have

$$u^2 y'' + uy' - y = 0$$

with characteristic equation

$$\begin{aligned} \lambda(\lambda-1)+\lambda-1 &= 0 \\ \lambda^2 - 1 &= 0 \\ \lambda &= \pm 1 \end{aligned}$$

which means the solutions are directly equivalent to the solutions found above:

$$\begin{aligned} y &= u, \frac{1}{u} \\ &= x-1, \frac{1}{x-1} \end{aligned}$$

4.6.2 Frobenius series solutions

So how can we modify the Taylor form of a solution $y(x)$ for the general case that a singular point exists? It turns out that there are at most two modifications needed to find such a solution. The main concept is to allow non-integer powers for the terms in the series, an idea best introduced with an example:

Example 4.26. Consider

$$2xy'' + 3y' - y = 0$$

This equation has a regular singular point at $x = 0$:

$$2\left(x^2 y''\right) + 3\left(xy'\right) - xy = 0$$

Hence let us try a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma} = x^{\sigma} \sum_{n=0}^{\infty} a_n x^n$$

Note that this is *not* a Taylor series, but instead a *Frobenius series*. We will need to determine the new index σ as well as the coefficients a_n - in fact, to avoid the problem of having $x + x^2 + \cdots$ and $x(1 + x + \cdots)$ as distinct solutions, we will require $a_0 \neq 0$, so that σ is fixed to be the first power to appear with a non-zero coefficient in the series expansion.

We continue as before, remembering that we have to differentiate with σ in the power, so that we have

$$
\begin{aligned}
\sum_{n=0}^{\infty}\left[2\left(n+\sigma\right)\left(n+\sigma-1\right)+3\left(n+\sigma\right)-x\right]a_n x^{n+\sigma} &= 0\\
\sum_{n=0}^{\infty}\left[\left[\left(n+\sigma\right)\left(2n+2\sigma+1\right)\right]a_n-a_{n-1}\right]x^{n+\sigma} &= 0
\end{aligned}
$$

This gives us the general recurrence relation for the problem:

$$
\boxed{\left[\left(n+\sigma\right)\left(2n+2\sigma+1\right)\right]a_n-a_{n-1}=0}
$$

Now $n=0$ gives the so-called *indicial equation*, from the requirement that $a_0\neq 0$ and the fact that $a_{-m}=0$ for any m:

$$
\begin{aligned}
\left(0+\sigma\right)\left(0+2\sigma+1\right) &= 0\\
\sigma\left(2\sigma+1\right) &= 0\\
\sigma &= -\frac{1}{2},0
\end{aligned}
$$

This allows us to split the analysis into the two cases according to these two roots:

$\sigma=-\frac{1}{2}$: We have the recurrence relation

$$
a_n=\frac{a_{n-1}}{\left(n-\frac{1}{2}\right)\cdot 2n}=\frac{a_{n-1}}{\left(2n-1\right)\cdot n}
$$

so for a given a_0 we have

$$
\begin{aligned}
a_1 &= \frac{a_0}{1\cdot 1}\\
a_2 &= \frac{a_1}{3\cdot 2}=\frac{a_0}{\left(3\cdot 1\right)\cdot\left(2\cdot 1\right)}\\
a_3 &= \frac{a_2}{5\cdot 3}=\frac{a_0}{\left(5\cdot 3\cdot 1\right)\cdot\left(3\cdot 2\cdot 1\right)}\\
a_n &= \frac{a_0}{\left(2n-1\right)\left(2n-3\right)\cdot\cdots\cdot 5\cdot 3\cdot 1\cdot n!}\\
&= \frac{a_0}{\left[(2n)!/\left(2n\cdot\left(2n-2\right)\cdot\cdots\cdot 4\cdot 2\right)\right]\cdot n!}\\
&= \frac{a_0}{\left[(2n)!/\left(2^n n!\right)\right]\cdot n!}\\
&= \frac{2^n a_0}{(2n)!}
\end{aligned}
$$

This gives the solution

$$\begin{aligned} y_1 &= a_0 x^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{2^n x^n}{(2n)!} \\ &= a_0 x^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{(2x)^n}{(2n)!} \\ &= a_0 x^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{\left(\sqrt{2x}\right)^{2n}}{(2n)!} \\ &= a_0 x^{-\frac{1}{2}} \cosh\left(\sqrt{2x}\right) \end{aligned}$$

which we are lucky enough to be able to write in closed form.

$\sigma = 0$: This solution has recurrence relation

$$a_n = \frac{a_{n-1}}{n\,(2n+1)}$$

and the solution is, similarly to the above, given by

$$\begin{aligned} a_n &= \frac{a_0}{n!\,(2n+1)!/\,(2^n n!)} \\ &= \frac{2^n a_0}{(2n+1)!} \end{aligned}$$

and hence

$$\begin{aligned} y_2 &= a_0 x^0 \sum_{n=0}^{\infty} \frac{2^n x^n}{(2n+1)!} \\ &= a_0 \sum_{n=0}^{\infty} \frac{\left(\sqrt{2x}\right)^{2n}}{(2n+1)!} \\ &= \frac{a_0}{\sqrt{2x}} \sum_{n=0}^{\infty} \frac{\left(\sqrt{2x}\right)^{2n+1}}{(2n+1)!} \\ &= \frac{a_0}{\sqrt{2}} x^{-\frac{1}{2}} \sinh\left(\sqrt{2x}\right) \end{aligned}$$

which again is fortunately amenable to a closed form representation.

The general solution is therefore

$$y = x^{-\frac{1}{2}} \left[A \cosh\sqrt{2x} + B \sinh\sqrt{2x}\right]$$

Now we can run into difficulties when the roots of the indicial equation are separated by an integer, because it is possible that the second solution gets ‘caught up’ in the first, in some sense. Here are two examples, the first of which shows that it is still possible that the basic approach works, and the second of which shows how it can totally fail.

Example 4.27. Consider

$$9y'' - \frac{6}{x}y' + \left(9 + \frac{4}{x^2}\right) y = 0$$

This equation has a regular singular point at $x = 0$:

$$9\left(x^2 y''\right) - 6\left(xy'\right) + \left(9x^2 + 4\right) y = 0$$

Hence let us try a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma} = x^{\sigma} \sum_{n=0}^{\infty} a_n x^n$$

This gives us the equation

$$\begin{aligned} \sum_{n=0}^{\infty} \left[9\left(n+\sigma\right)\left(n+\sigma-1\right) - 6\left(n+\sigma\right) + 9x^2 + 4\right] a_n x^{n+\sigma} &= 0 \\ \sum_{n=0}^{\infty} \left[\left[3\left(n+\sigma\right)\left(3n+3\sigma-5\right)+4\right] a_n + 9a_{n-2}\right] x^{n+\sigma} &= 0 \end{aligned}$$

So the recurrence relation is

$$\boxed{\left[3\left(n+\sigma\right)\left(3n+3\sigma-5\right)+4\right] a_n + 9a_{n-2} = 0}$$

The indicial equation is easily solved:

$$\begin{aligned} 3\left(0+\sigma\right)\left(0+3\sigma-5\right)+4 &= 0 \\ 9\sigma^2 - 15\sigma + 4 &= 0 \\ \left(3\sigma-1\right)\left(3\sigma-4\right) &= 0 \\ \sigma &= \frac{1}{3}, \frac{4}{3} \end{aligned}$$

Now we have two roots separated by an integer, namely 1. Let us see what happens:

$\sigma = \frac{1}{3}$: We have the recurrence relation

$$a_n = -\frac{9a_{n-2}}{\left(3n+1\right)\left(3n-4\right)+4} = -\frac{a_{n-2}}{n\left(n-1\right)}$$

so for even n, we have

$$a_{2m} = \left(-1\right)^m \frac{a_0}{(2m)!}$$

and for odd n we have an arbitrary a_1 followed by

$$a_{2m+1} = \left(-1\right)^m \frac{a_1}{(2m+1)!}$$

$\sigma = \frac{4}{3}$: Already implicitly taken account of by the freedom in choice of a_1.

The general solution can therefore be written as

$$y = x^{\frac{1}{3}} \left[a_0 \cos x + a_1 \sin x\right]$$

This case worked out to give us two solutions; however, this does not necessarily have to happen:

Example 4.28. Consider

$$x^2 y'' + xy' + \left(x^2 - k^2\right) y = 0$$

where $k \geq 0$ is an integer.

This equation manifestly has a regular singular point at $x = 0$. So guess

$$y = \sum_{n=0}^{\infty} a_n x^{n+\sigma}$$

Thus we have

$$\begin{aligned} \sum_{n=0}^{\infty} \left[(n+\sigma)(n+\sigma-1) + (n+\sigma) + x^2 - k^2\right] a_n x^{n+\sigma} &= 0 \\ \sum_{n=0}^{\infty} \left[\left[(n+\sigma)^2 - k^2\right] a_n + a_{n-2}\right] x^{n+\sigma} &= 0 \end{aligned}$$

So the recurrence relation is

$$\boxed{\left[(n+\sigma)^2 - k^2\right] a_n + a_{n-2} = 0}$$

Solving the indicial equation gives us the values for σ:

$$\begin{aligned} (0+\sigma)^2 - k^2 &= 0 \\ \sigma &= \pm k \end{aligned}$$

Again, we have two roots separated by an integer, $2k$. Let us see what happens:

$\sigma = k$: We have the recurrence relation

$$a_n = -\frac{a_{n-2}}{n(n+2k)}$$

so for even n, we have

$$\begin{aligned} a_{2m} &= (-1)^m a_0 \frac{1}{2m(2m-2)\cdots 4\cdot 2 \times (2m+2k)(2m+2k-2)\cdots(2+2k)\cdot 2k} \\ &= (-1)^m a_0 \frac{k!}{2^m (m)! 2^m (m+k)!} \\ &= a_0 k! (-1)^m \frac{1}{2^{2m} (m)! (m+k)!} \end{aligned}$$

and for odd n we have $a_n = 0$ since $a_1 = a_{-1}/(1+2k) = 0$, and future values are all easily seen from this as being zero.

Thus, rescaling slightly by $1/\left(k!2^k\right)$, we have the solution

$$y_1 = a_0' J_k(x) \equiv a_0' x^k \sum_{m=0}^{\infty} \frac{(-1)^m}{2^{2m+k} m!(m+k)!} x^{2m}$$

($J_m(x)$ is called a *Bessel function of the first kind*)

$\sigma = -k$: This time, we have

$$a_n = -\frac{a_{n-2}}{n(n-2k)}$$

which again gives $a_n = 0$ for odd n. However, for even n something surprising happens: the formula becomes invalid at $n = 2k$. Let us go back, then, to the original recurrence relation $\left[(n+\sigma)^2 - k^2\right] a_n + a_{n-2} = n(n-2k)a_n + a_{n-2} = 0$. At $n = 2k$ this tells us

$$\begin{aligned} 0a_n + a_{n-2} &= 0 \\ a_{n-2} &= 0 \end{aligned}$$

But then $a_{n-4} = -(n-2)(n-2-2k)a_{n-2} = 0$, and so on, all the way back to $a_2 = 0$. But then

$$2 \cdot (2-2k)a_2 + a_0 = 0$$

implies that $a_0 = 0$, which is a contradiction! (Recall we originally choose σ so that a_0 was the least *non-zero* coefficient.)

Hence there is no general solution of this form, as one degree of freedom is insufficient to match any initial conditions.

So what are we missing? To work out the form of the next solution, we can apply the techniques we developed in section 4.1.3 on the method of reduction of order.

$$\begin{aligned} y_2 &= vy_1 \\ y'' + py' + qy &= 0 \\ (v''y_1 + 2v'y_1' + vy_1'') + p(v'y_1 + vy_1') + qvy_1 &= 0 \\ v[y_1'' + py_1' + qy_1] + v''y_1 + (2y_1' + py_1)v' &= 0 \\ v''y_1 + (2y_1' + py_1)v' &= 0 \end{aligned}$$

From this equation we can immediately solve for v':

$$\begin{aligned}\frac{v''}{v'} &= -\frac{2y_1' + py_1}{y_1}\\ \ln|v'| &= -2\ln|y_1| - \int p\mathrm{d}x\\ v' &= \frac{1}{y_1^2}e^{-\int p\mathrm{d}x}\end{aligned}$$

Then it follows that there is a solution

$$y_2 = y_1 \int \frac{1}{y_1^2}e^{-\int p\mathrm{d}x}\mathrm{d}x$$

or, writing this more carefully,

$$y_2(x) = y_1(x)\int^x \frac{1}{y_1^2(t)}e^{-\int^t p(u)\mathrm{d}u}\mathrm{d}t$$

which is a result that can also be derived from using the Wronskian $W = W_0 e^{-\int p\mathrm{d}x}$.

Now assume we already have a series solution for $y_1(x)$ which we will write as

$$y_1 = x^{\sigma_1}\sum_{n=0}^{\infty} a_n x^n$$

taking $x_0 = 0$ for simplicity, and that we also have a Taylor series for $xp(x)$ that we write

$$p = \frac{\alpha_{-1}}{x} + \sum_{n=0}^{\infty}\alpha_n x^n$$

and $x^2 q(x)$ that we write

$$q = \frac{\beta_{-2}}{x^2} + \sum_{n=0}^{\infty}\beta_n x^n$$

Now the indicial equation can be calculated in terms of these series by expanding xp and x^2q as follows:

$$\begin{aligned}\left(x^2y''\right) + (xp)(xy') + \left(x^2q\right)y &= 0\\ \sum_{n=0}^{\infty}\left[(n+\sigma)(n+\sigma-1) + xp(x)(n+\sigma) + x^2q(x)\right]a_n x^{n+\sigma} &= 0\\ \sigma(\sigma-1) + \alpha_{-1}\sigma + \beta_{-2} &= 0\\ \sigma^2 + (\alpha_{-1}-1)\sigma + \beta_{-2} &= 0\end{aligned}$$

Hence the sum of the two roots is $\sigma_1 + \sigma_2 = 1 - \alpha_{-1}$, and we already know that $\sigma_1 - \sigma_2 = m \geq 0$ is a non-negative integer (as σ_1 was assumed to be the larger root, and they differ by an integer). It follows that $2\sigma_1 = 1 - \alpha_{-1} + m$ so that

$$y_1^2(t) = \left(t^{\sigma_1}\right)^2 \left[\sum_{n=0}^{\infty} a_n t^n\right]^2 = t^{2\sigma_1} \left[\sum_{n=0}^{\infty} a_n t^n\right]^2 = t^{(1-\alpha_{-1})+m} \sum_{n=0}^{\infty} a'_n t^n$$

for some other set of coefficients a'_n - note that $a'_0 \neq 0$ still holds. The other expression we need a Taylor series for is $\exp\left(-\int^t p\mathrm{d}u\right)$:

$$\begin{aligned}
\exp\left(-\int^t p\mathrm{d}u\right) &= \exp\left(-\int^t \left[\frac{\alpha_{-1}}{u} + \sum_{n=0}^{\infty} \alpha_n u^n\right] \mathrm{d}u\right) \\
&= \exp\left(-\alpha_{-1} \int^t \frac{\mathrm{d}u}{u}\right) \exp\left(-\sum_{n=0}^{\infty} \alpha'_n t^n\right) \\
&= t^{-\alpha_{-1}} \sum_{n=0}^{\infty} \alpha''_n t^n
\end{aligned}$$

for new coefficients α'_n and α''_n, noting that $\exp(f(x)) = 1 + f(x) + f(x)^2/2 + \cdots$ is everywhere equal to its power series for the second term, and recalling the results on $e^{\int \frac{\mathrm{d}x}{x}}$ discussed in section 3.1 for the first. (The constant term from the integral is absorbed into the α''_n.)

We are now ready to construct y_2:

$$\begin{aligned}
y_2(x) &= y_1(x) \int^x \frac{1}{y_1^2(t)} e^{-\int^t p(u)\mathrm{d}u} \mathrm{d}t \\
&= y_1(x) \int^x \left[t^{(1-\alpha_{-1})+m} \sum_{n=0}^{\infty} a'_n t^n\right]^{-1} t^{-\alpha_{-1}} \sum_{n=0}^{\infty} \alpha''_n t^n \mathrm{d}t \\
&= y_1(x) \int^x t^{-1-m} \left[\sum_{n=0}^{\infty} a'_n t^n\right]^{-1} \sum_{n=0}^{\infty} \alpha''_n t^n \mathrm{d}t
\end{aligned}$$

Now recall that $a'_0 \neq 0$, so that the ratio of the two Taylor series itself has a Taylor series around $t = 0$ (as the denominator has a non-zero value a'_0 at this point); hence we can write

$$y_2 = y_1 \int^x t^{-1-m} \sum_{n=0}^{\infty} \gamma_n t^n \mathrm{d}t$$

Now assuming that we can integrate the series term by term[9] we get a few initial terms integrating negative powers of t, then another Taylor series. The reason that a special case arises for integer m is that we can have (though we do not necessarily have to) a term $t^{-1-m}\gamma_m t^m = \gamma_m/t$ which does not integrate to give a polynomial:

$$y_2 = y_1 \left[x^{-m} \sum_{n=0}^{\infty} \gamma'_n x^n + \gamma_m \ln x\right]$$

[9]This is valid; there are a few initial terms with negative powers of t, and then a Taylor series. The integrated expression for a Taylor series has the same radius of convergence as the original.

The general solution has a ln-type singularity at $x = 0$!

Note that by recalling the expression for y_1 as a power series, we can incorporate this into the new Taylor series via

$$\begin{aligned} y_2 &= \gamma_m y_1 \ln x + x^{-m} \left[x^{\sigma_1} \sum_{n=0}^{\infty} a_n x^n \right] \left[\sum_{n=0}^{\infty} \gamma'_n x^n \right] \\ &= \gamma_m y_1 \ln x + x^{\sigma_1 - m} \left[\sum_{n=0}^{\infty} b_n x^n \right] \\ &= \gamma_m y_1 \ln x + x^{\sigma_2} \left[\sum_{n=0}^{\infty} b_n x^n \right] \end{aligned}$$

Indeed, it turns out that the remaining solution is in general of the form

$$\boxed{y_2 = \ln(x - x_0)\, y_1 + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+\sigma_2}}$$

where $y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma_1}$ is the normal solution corresponding to the *larger* solution $\sigma_1 \geq \sigma_2$.

> **Example 4.29.** The second solution to $x^2 y'' + x y' + \left(x^2 - k^2\right) y = 0$ where $k \geq 0$ is an integer is of the form
>
> $$y_2 = \ln x J_k(x) + \sum_{n=0}^{\infty} b_n x^{n-k}$$

There is in fact a general theorem about the existence of these solutions[10], which we are not going to prove formally here.

Theorem 4.30 (Fuchs). *If $x = x_0$ is a regular singular point, then there is at least one solution to the equation $y'' + py' + qy = 0$ of the form*

$$\boxed{y_1 = \sum_{n=0}^{\infty} a_n (x - x_0)^{n+\sigma}}$$

and any other solution is either of the same form, or of the form

$$\boxed{y_2 = \ln(x - x_0)\, y_1 + \sum_{n=0}^{\infty} b_n (x - x_0)^{n+\tau}}$$

where y_1 is a solution of the first form. The power series have a radius of convergence at least as large as those of the relevant coefficients.

[10] See "Lectures on differential and integral equations" by Kōsaku Yosida.

As a final observation, note that if we have a forced equation

$$y'' + p(x)\, y' + q(x)\, y = g(x)$$

then solving this for power series is precisely equivalent to finding a particular solution and then solving the homogeneous version of the equation; as a result, essentially the same ideas apply, if we assume that we can find a particular solution.

4.7 Systems of Linear Equations

Another case we have conspicuously failed to address so far is that where we have more than one unknown variable. Consider, for example, the two first-order equations

$$\begin{aligned} \dot{y}_1 &= a y_1 + b y_2 + f_1(t) \\ \dot{y}_2 &= c y_1 + d y_2 + f_2(t) \end{aligned}$$

We can write this more concisely in terms of a vector $\mathbf{Y}$, using yet another vector notation:

$$\dot{\mathbf{Y}} = M\mathbf{Y} + \mathbf{F}$$

where

$$\mathbf{Y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \mathbf{F} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$$

More generally, $M = M(t)$ can be a function of the independent variable, if we wish to encode equations without constant coefficients. However, we will leave these ideas until section 6, where we will discuss the general solution of these equations.

4.7.1 Equivalence to higher-order equations

The first notable thing we can do is perhaps suggested by the vector formulation of the problem - it is clear that the case where $\mathbf{Y}$ is a solution vector with entries y and $\dot{y}$ corresponds to something like a second-order equation. In fact, let us take an equation

$$\ddot{y} + a\dot{y} + by = f$$

and write $y_1 = y$, $y_2 = \dot{y}$. Then $\dot{y} = y_2$ and $\dot{y}_2 = \ddot{y} = f - a y_2 - b y_1$ so

$$\dot{\mathbf{Y}} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} \mathbf{Y} + \begin{pmatrix} 0 \\ f \end{pmatrix}$$

As we will see in the later section, we can easily extend this to write any nth-order ODE as a system of n first-order ODEs.

Strikingly, though, it is also possible to reverse this reduction, and transform a two-variable first-order problem into a one-variable second-order problem. Consider the value of $\ddot{y}_1$:

$$\begin{aligned}
\ddot{y}_1 &= a\dot{y}_1 + b\dot{y}_2 + \dot{f}_1 \\
&= a\dot{y}_1 + (bcy_1 + bdy_2 + bf_2) + \dot{f}_1 \\
&= a\dot{y}_1 + bcy_1 + d(\dot{y}_1 - ay_1 - f_1) + bf_2 + \dot{f}_1 \\
\ddot{y}_1 - (a+d)\,\dot{y}_1 + (ad-bc)\,y_1 &= bf_2 - df_1 + \dot{f}_1
\end{aligned}$$

Note that the homogeneous version of this equation is simply

$$\ddot{y}_1 - \operatorname{tr}(M)\,\dot{y}_1 + \det(M)\,y_1 = 0$$

where tr and det are the trace and determinant respectively - in fact, for this constant coefficient case, this has exactly the same characteristic equation as the matrix! Therefore, the eigenvalues of the matrix are precisely the eigenvalues of the differential operator here.

Once y_1 is known, finding y_2 is trivial from the original equations (although if $b = 0$ we have to solve the second first-order ODE for y_2). We note that y_2 should have solutions which are multiples of the same eigenfunctions.

4.7.2 Solving the system

The fact that the solutions for y_1 are in general of the form $e^{\lambda t}$ (and the related eigenfunctions) where λ is an eigenvalue of the matrix suggests that we may be able to solve this problem in a very similar way to the one-dimensional case, by looking for constant multiples of the complementary function.

Consider

$$\begin{aligned}
\dot{\mathbf{Y}} &= M\mathbf{Y} + \mathbf{F} \\
\dot{\mathbf{Y}} - M\mathbf{Y} &= \mathbf{F}
\end{aligned}$$

We guess that there is a complementary function of the form

$$\mathbf{Y}_c = \mathbf{v}e^{\lambda t}$$

noting that the equation is still linear. This implies that

$$\begin{aligned}
\lambda\mathbf{v}e^{\lambda t} - M\mathbf{v}e^{\lambda t} &= 0 \\
M\mathbf{v} &= \lambda\mathbf{v}
\end{aligned}$$

which is precisely the statement that $\mathbf{v}$ is an eigenvalue of the matrix M with eigenvalue λ, which fits in very well from what we noted above.

Recall that we can obtain the characteristic equation for matrix eigenvalue λ by noting that $(M - \lambda I)\,\mathbf{v} = \mathbf{0}$ for some $\mathbf{v} \neq \mathbf{0}$, and hence $\det(M - \lambda I) = 0$. Then we must find the appropriate vector $\mathbf{v}$.

Remark. We will not address the case of a repeated eigenvalue, where we may not even have two eigenvectors, or of forcing proportional to an eigenvalue, in this section. The generalizations from the single-variable case are fairly direct, and we will give a more complete treatment in section 6.3.

Example 4.31. Solve

$$\dot{\mathbf{Y}} = \begin{pmatrix} -2 & 1 \\ 15 & -4 \end{pmatrix} \mathbf{Y}$$

Consider the trial complementary function $\mathbf{Y}_c = \mathbf{v}e^{\lambda t}$. Then we have

$$\begin{aligned} \begin{vmatrix} -2-\lambda & 1 \\ 15 & -4-\lambda \end{vmatrix} &= 0 \\ (2+\lambda)(4+\lambda) - 15 &= 0 \\ \lambda^2 + 6\lambda - 7 &= 0 \\ (\lambda - 1)(\lambda + 7) &= 0 \end{aligned}$$

so the eigenvalues are $\lambda = 1, -7$. We must find the eigenvector for each case:

$\lambda = 1$: We see $(M - \lambda I)\mathbf{v} = \mathbf{0}$, so

$$\begin{aligned} \begin{pmatrix} -3 & 1 \\ 15 & -5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \mathbf{0} \\ -3v_1 + v_2 &= 0 \end{aligned}$$

and hence one solution is

$$\mathbf{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$\lambda = -7$: We have

$$\begin{aligned} \begin{pmatrix} 5 & 1 \\ 15 & 3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} &= \mathbf{0} \\ 5v_1 + v_2 &= 0 \end{aligned}$$

and so

$$\mathbf{v} = \begin{pmatrix} 1 \\ -5 \end{pmatrix}$$

So the general complementary function (our general solution) can be written as

$$\begin{aligned} \mathbf{Y} &= A\begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{t} + B\begin{pmatrix} 1 \\ -5 \end{pmatrix} e^{-7t} \\ y_1 &= Ae^{t} + Be^{-7t} \\ y_2 &= 3Ae^{t} - 5Be^{-7t} \end{aligned}$$

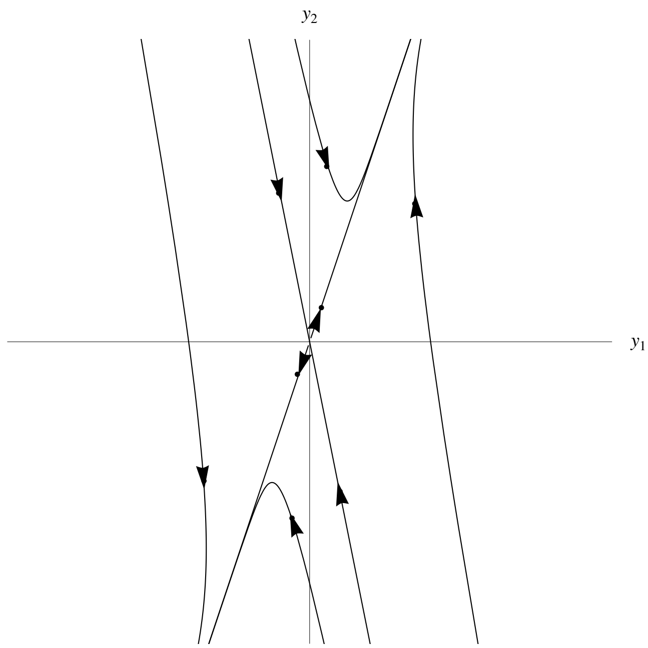


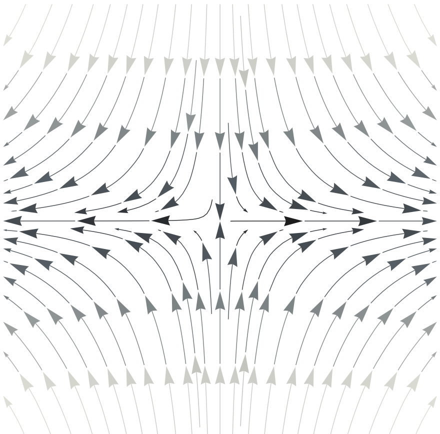
Figure 4.13: The trajectories in phase space of the example

Just as we could draw a two-dimensional phase-space for second order equations, with variables y_1 and $\dot{y}_1$, it is possible to depict all behaviour of systems like this on a diagram with one axis corresponding to y_1 and another corresponding to y_2. Figure 4.13 shows that the system has a saddle point at $(0, 0)$, and hence this point is overall unstable, because an initial point with a small component in the $(1, 3)$ direction will tend to $\pm\infty$.

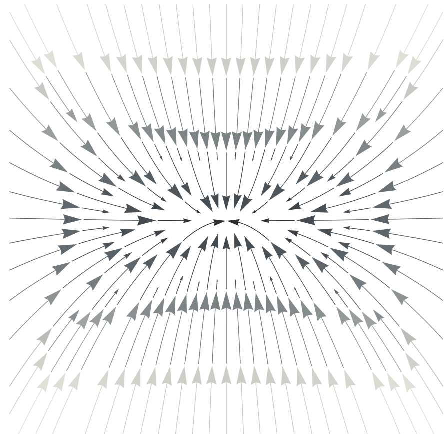
In general, if

$$\mathbf{Y} = \mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t}$$

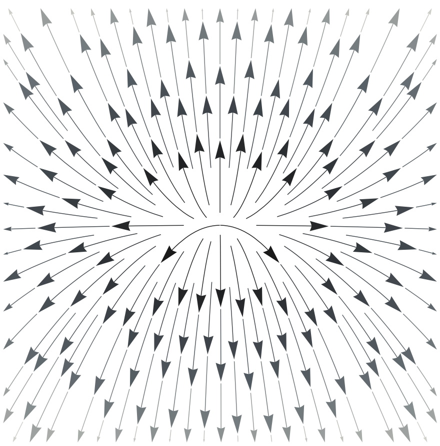
then we categorize the point $(0, 0)$ as follows, ignoring the degenerate cases with repeated roots (for examples, see Figure 4.14):



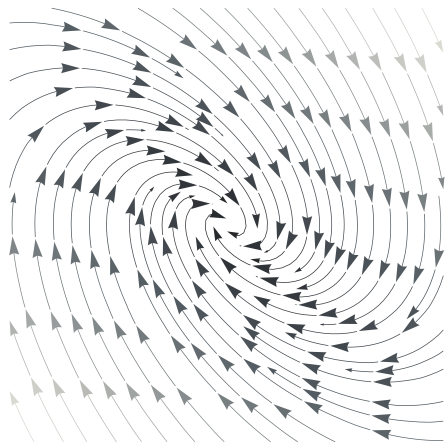
(a) Saddle



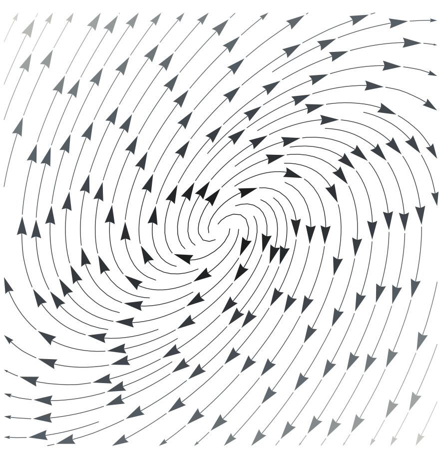
(b) Stable node, or nodal sink



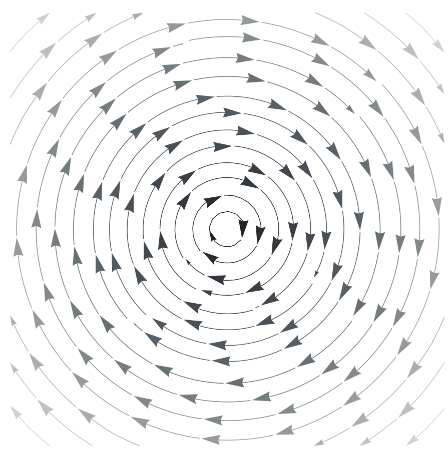
(c) Unstable node, or nodal source



(d) Stable spiral, or spiral sink



(e) Unstable spiral, or spiral source



(f) Centre

Figure 4.14: Flow diagrams for non-degenerate solutions

(i) If λ_1, λ_2 are both real, and one is positive and the other negative, then we get a *saddle* as was shown in our example Figure 4.13.

(ii) If they are both real and have the same sign, with $\lambda_1 \neq \lambda_2$, then either

(a) the roots are both negative, and we get a stable node, since as $t \to \infty$, any initial $\mathbf{Y}_0$ tends to $\mathbf{0}$.

(b) the roots are both positive, and we get an unstable node, which is the same with the direction ('time') reversed.

(iii) If λ_1 and λ_2 are complex, then since the system is real, they are complex conjugates. The behaviour of the system in then determined by the magnitude of the real part of the roots.

(a) If $\operatorname{Re}(\lambda) < 0$ then we get a *stable spiral*, since the magnitude decays, whilst the direction oscillates.

(b) If $\operatorname{Re}(\lambda) > 0$ then we get an *unstable spiral*, which is the reversed version of the stable spiral.

(c) In the special case of a pure imaginary pair of roots λ, we get a *centre*, in which the magnitude never changes, but we still get oscillation - hence the system is entirely periodic.

Example 4.32. Now consider the following forced version of the above equation:

$$\dot{\mathbf{Y}} = \begin{pmatrix} -2 & 1 \\ 15 & -4 \end{pmatrix} \mathbf{Y} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{2t}$$

We can make the educated guess (based on the fact that this is exactly the same as forcing a second-order system) that there is a particular solution

$$\mathbf{Y} = \mathbf{u} e^{2t}$$

so that

$$2 \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ 15 & -4 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

which implies that

$$\begin{pmatrix} 4 & -1 \\ -15 & 6 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

and we can invert this matrix (since 2 is not an eigenvalue, we know this matrix will always be invertible) to get

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} &= \frac{1}{9} \begin{pmatrix} 6 & 1 \\ 15 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \\ &= \frac{1}{9} \begin{pmatrix} 13 \\ 34 \end{pmatrix} \end{aligned}$$

Hence this system has a general solution

$$\begin{aligned}
\mathbf{Y} &= A\begin{pmatrix}1\\3\end{pmatrix}e^{t} + B\begin{pmatrix}1\\-5\end{pmatrix}e^{-7t} - \frac{1}{9}\begin{pmatrix}13\\34\end{pmatrix}e^{2t}\\
y_1 &= Ae^{t} + Be^{-7t} - \frac{13}{9}e^{2t}\\
y_2 &= 3Ae^{t} - 5Be^{-7t} - \frac{34}{9}e^{2t}
\end{aligned}$$

5 Partial Differential Equations

The final new topic we will discuss in this course is the field of *partial differential equations*. This is a very important field which is very poorly understood in general. Partial differential equations often arise in physical systems where the rate of change of some quantity over time is dependent on its rate of change in space - for example, transfers of heat occur more rapidly when there is a larger heat gradient, so there are partial derivatives with respect to t and x in the one-dimensional heat equation. We will study the so-called *diffusion equation* below.

However, first we will consider a more fundamental idea which arises with incredible frequency in physical problems: the wave equation.

5.1 Wave Equations

We will first consider the simplest construction of an abstract wave equation.

5.1.1 First-order wave equation

Imagine some quantity $y(x,t)$ which oscillates in the presence of waves passing through the medium at a constant speed c. If we pick a point of fixed height on a propagating wave, $x_1(t)$, then we have

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \pm c$$

It follows that, using the chain rule,

$$\begin{aligned} \frac{\partial y(x_1(t),t)}{\partial t} &= 0 \\ \frac{\partial y(x_1,t)}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial y(x_1,t)}{\partial t} &= 0 \\ \frac{\partial y}{\partial t} &= \mp c\frac{\partial y}{\partial x} \end{aligned}$$

The equation

$$\boxed{\frac{\partial y}{\partial t} = c\frac{\partial y}{\partial x}}$$

is an *advection equation*, which can be described as the (unforced) *first-order wave equation* for y, where c is the constant speed of wave propagation, which we have constrained to be in the *negative* x direction as time passes by choosing a sign.

Since $\mathrm{d}y/\mathrm{d}t = 0$ along paths in (t,x) space with $\mathrm{d}x/\mathrm{d}t = -c$, namely $x = x_0 - ct$, we can always write

$$y = f(x_0) = f(x+ct)$$

In fact, since this does indeed solve the equation even without any information about f, this is the *general solution* of the first-order wave equation. Note that we do not have an unknown constant, but instead an *unknown function*. This is characteristic of partial differential equations. Another way of looking at it is that we need one initial condition for every single $x+ct$ paths - a continuous infinity of initial conditions.

Remark. Of course, in order for the equation to be well-defined we need f to have the first partial derivatives in x and t involved in the equation, so f is not totally unconstrained unless we allow some sort of singular behaviour.

The result is, of course, not surprising given how we derived it. We thought about a wave-like shape (for example, a *wavelet*, or a bounded oscillation) traveling along at a constant velocity of $-c$, and we found that the solutions consist of all functions which move left at that speed over time, as shown in Figure 5.1.

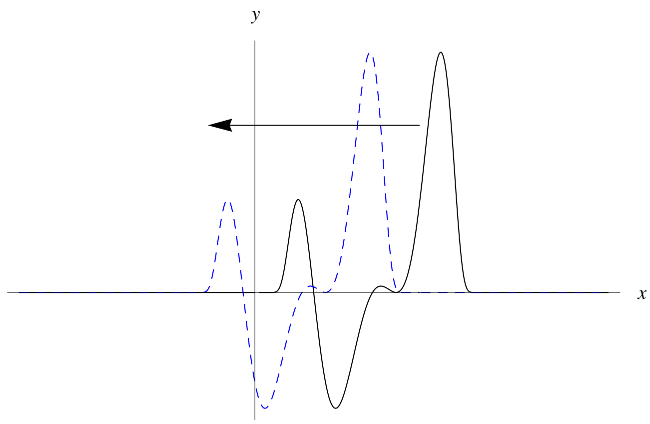


Figure 5.1: The translation of an example wavelet at speed c

Explicitly, if the initial condition if $y = F(x)$ at time $t = 0$, then $y = F(x + ct)$ at later times:

$$y(x, 0) = F(x) \quad \Longrightarrow \quad y(x, t) = F(x + ct)$$

Example 5.1. Solve

$$\frac{\partial y}{\partial t} = c\frac{\partial y}{\partial x} \quad \text{with} \quad y(x, 0) = x^2 - 3$$

This is now trivial to solve, since we know $y = f(x + ct)$, and hence using the initial condition, we immediately have

$$\begin{aligned} x^2 - 3 &= f(x + 0) \\ y &= (x + ct)^2 - 3 \end{aligned}$$

Note that the paths in (t, x) space along which y is constant are by definition exactly the *contours* of y in this space, as shown in Figure 5.2. Along each one these curves, y is reduced to a simple ordinary differential equation - in this case the simple $y' = 0$ - with an independent set of initial conditions for each curve. These curves are called *characteristic curves* or just *characteristics*, and the method of characteristics solves PDEs by finding these curves and then solving the resulting families of ODEs.

The method of characteristics clearly also works for PDEs with inhomogeneities, so long as we can solve the associated ODE. For instance:

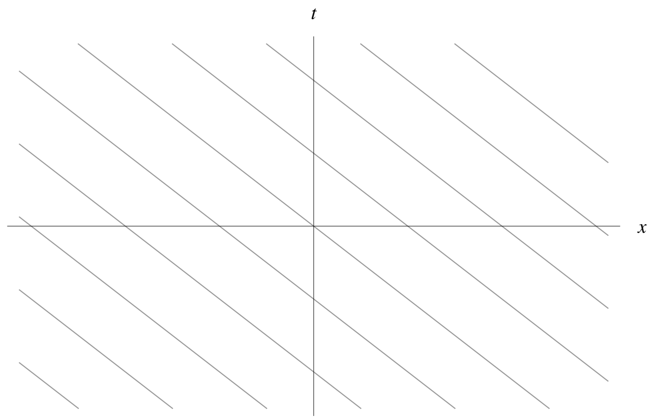


Figure 5.2: The form of contours of any solution to the first order wave equation

Example 5.2. Solve

$$\frac{\partial y}{\partial t} = c\frac{\partial y}{\partial x} + \omega \sin(\omega t) \quad \text{with} \quad y(x,0) = \cos(\rho x)$$

In this case, along the characteristic curves

$$\begin{aligned} \frac{\mathrm{d}x}{\mathrm{d}t} &= -c \\ \implies x &= x_0 - ct \end{aligned}$$

we have

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}t} &= \omega \sin(\omega t) \\ \implies y(t) &= A - \cos(\omega t) \end{aligned}$$

Now at time $t = 0$ we have

$$y(x,0) = A - 1 = \cos(\rho x)$$

and since $x(0) = x_0$ it follows that

$$\begin{aligned} A &= 1 + \cos(\rho x_0) \\ y(x,t) &= 1 + \cos(\rho [x + ct]) - \cos(\omega t) \end{aligned}$$

5.1.2 Second-order wave equation

A more physically derived version of the wave equation can be derived by taking the physical limit of a suitable discrete medium, such as a series of springs connected to each other with masses.

Imagine a series of identical small masses m connected in a line by identical springs with spring constant k of natural length h. The equilibrium positions of three adjacent masses are $x - h$, x and

$x+h$ - let $u(x)$ be the function giving the distance from the equilibrium position of the mass normally located at x.

Then Newton's second law says that the force is related to the acceleration by

$$F = m\ddot{u}(x,t) = m\frac{\partial^2 u}{\partial t^2}$$

whilst Hooke's law states that

$$\begin{aligned} F &= F_{x+h} + F_{x-h} \\ &= k\left[u(x+h,t) - u(x,t)\right] + k\left[u(x-h,t) - u(x,t)\right] \\ &= k\left[u(x+h,t) + u(x-h,t) - 2u(x,t)\right] \end{aligned}$$

Equating these two forces gives the following equation of motion:

$$m\frac{\partial^2 u}{\partial t^2} = k\left[u(x+h,t) - 2u(x,t) + u(x-h,t)\right]$$

To take the limit of $h \to 0$, we consider $N \to \infty$ masses, spaced along a constant length $L = Nh$ and weighing a constant total mass $M = Nm$ - the total stiffness of the spring is also kept constant at $K = k/N$. Then the above equation can be rewritten as

$$\frac{\partial^2 u}{\partial t^2} = \frac{KL^2}{M} \cdot \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}$$

and the limit can now be taken easily:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = \frac{KL^2}{M}\frac{\partial^2 u}{\partial x^2}}$$

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}}$$

This equation can also be written

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$$

and is hence called a *hyperbolic*[11] partial differential equation, entirely by analogy with the implicit formula for a hyperbola: $x^2/a^2 - y^2/b^2 = 1$, though note that it is homogeneous.

In fact, this homogeneity makes it tempting to solve the equation by 'factoring' the differential operator:

$$\underbrace{\left(\frac{\partial}{\partial x} - c\frac{\partial}{\partial t}\right)}_{D_1}\underbrace{\left(\frac{\partial}{\partial x} + c\frac{\partial}{\partial t}\right)}_{D_2} u = D_2 D_1 u = 0$$

[11] Linear, second-order PDEs are classified in broadly the same way as polynomials are, but these definitions can be extended via more abstract conditions to higher order equations.

and deducing that u satisfies the original equation if u satisfies one of the two advection (first-order wave) equations there. In fact, this is valid, because these two components *commute*, $D_1 D_2 \equiv D_2 D_1$:

$$\begin{aligned} u &= f(x+ct) \\ u &= g(x-ct) \end{aligned}$$

are both solutions, as you can easily check. Further, because of the linearity of the equation, so is any[12] function of the form

$$u = f(x+ct) + g(x-ct)$$

What is less clear is whether all solutions are of this form, or if we are missing some solutions. In order to check this, it is useful to apply a key idea in PDEs: a change of variables.

Theorem 5.3. *The second-order wave equation*

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

has the general solution

$$y = f(x+ct) + g(x-ct)$$

Proof. In fact, the fairly heuristic approach we have adopted suggests a natural change of variables, to

$$\begin{aligned} \xi &= x - ct \\ \eta &= x + ct \end{aligned}$$

Then

$$\begin{aligned} x &= \frac{1}{2}[\xi + \eta] \\ t &= \frac{1}{2c}[\eta - \xi] \end{aligned}$$

so

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{1}{2}\left[\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right] \\ \frac{\partial}{\partial t} &= \frac{1}{2c}\left[-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right] \end{aligned}$$

[12] As before, observe we need u to have the second partial derivatives involved.

which means that the original equation becomes

$$\begin{aligned}\frac{1}{4}\left[\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]^2 y &= \frac{1}{4}\left[-\frac{\partial}{\partial\xi}+\frac{\partial}{\partial\eta}\right]^2 y\\ \frac{\partial^2 y}{\partial\xi^2}+2\frac{\partial^2 y}{\partial\xi\partial\eta}+\frac{\partial^2 y}{\partial\eta^2} &= \frac{\partial^2 y}{\partial\xi^2}-2\frac{\partial^2 y}{\partial\xi\partial\eta}+\frac{\partial^2 y}{\partial\eta^2}\\ \frac{\partial^2 y}{\partial\xi\partial\eta} &= 0\end{aligned}$$

which we can solve straightforwardly by simply integrating, and remembering that 'constants' of integration become arbitrary functions of the variables held constant:

$$\begin{aligned}\frac{\partial y}{\partial\eta} &= f_1(\eta)\\ y &= f_2(\eta)+f_3(\xi)\end{aligned}$$

Therefore, the general solution is

$$y = f(x+ct)+g(x-ct)$$

for some arbitrary functions f and g. □

As before, we need infinitely many conditions in order to find f and g - for example, two pieces of information, $y(x,0)$ and $\partial y/\partial t$, as initial conditions (at $t=0$); and two boundary conditions, the value of y for some boundary points like $x\to\pm\infty$.

Example 5.4. Solve the (second-order) wave equation in the case

$$\begin{aligned}y(x,0) &= h(x)\\ \left[\frac{\partial y}{\partial t}\right](x,0) &= 0\end{aligned}$$

with the boundary conditions

$$y(x,t)\to 0 \quad \text{as} \quad x\to\pm\infty$$

and assuming that $h(x)\to 0$ as $x\to\pm\infty$.

This is the general case of an infinite medium held still in some shape $h(x)$ and then released at time $t=0$. From the initial conditions, we have

$$\begin{aligned}f(x)+g(x) &= h(x)\\ cf'(x)-cg'(x) &= 0\end{aligned}$$

Hence $f' = g'$, from which it follows that $f = g + C$. Thus

$$\begin{aligned} 2g(x) + C &= h(x) \\ g(x) &= \frac{h(x) - C}{2} \\ f(x) &= \frac{h(x) + C}{2} \end{aligned}$$

and so

$$\begin{aligned} y(x,t) &= f(x+ct) + g(x-ct) \\ &= \frac{h(x+ct) + C}{2} + \frac{h(x-ct) - C}{2} \\ &= \frac{h(x+ct) + h(x-ct)}{2} \end{aligned}$$

This general result shows that the result is that the shape of h splits into two identical components which travel in opposite directions at speed c.

Remark. We did not explicitly use the boundary conditions here, because the solution automatically obeys them if h does.

Example 5.5. If $h(x) = e^{-1/x^2}$ then

$$y(x,t) = \frac{1}{2}\left[e^{-1/(x+ct)^2} + e^{-1/(x-ct)^2}\right]$$

which is depicted in Figure 5.3 - the initial shape is the blue-shaded shape, gradually splitting into the shapes shown.

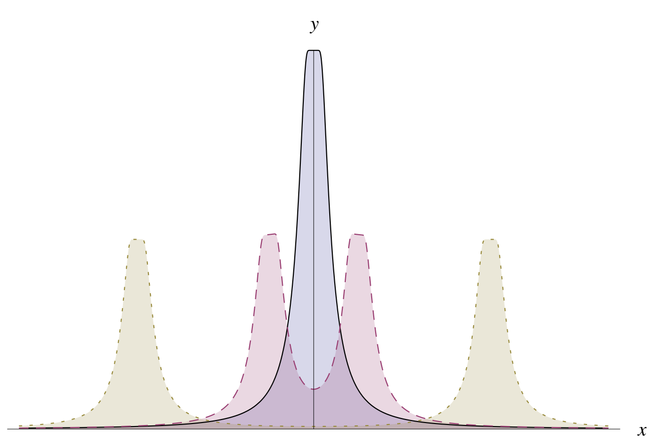


Figure 5.3: The separation of the initial packet into two identical components

5.2 Diffusion Equation

The final example we shall treat here demonstrates how partial differential equations can become rapidly more complicated to solve despite having extremely simple forms. It is motivated by the extremely physical problem of heat diffusion.

Let $T(x,t)$ be the temperature of a rod. Then the rate of change of the temperature is determined by the *diffusion equation*, which is one dimension is the PDE

$$\boxed{\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}}$$

where α is the *(thermal) diffusivity.* This is a constant for the problem of heat transfer, and in this case we call this equation the one dimensional *heat equation.* It is a *parabolic* PDE (to be compared with $y = x^2$, and contrasted to the hyperbolic wave equation).

Remark. The equation can be derived from Fourier's law, which states that the rate at which heat energy U flows is proportional to the (negative) temperature gradient across the boundary. Hence for any small period of time I_t and of space $I_x = [x_0 - \Delta x, x_0 - \Delta x]$, we have essentially

$$\begin{aligned} \int_{I_t}\int_{I_x} \frac{\partial U}{\partial t} \mathrm{d}x\mathrm{d}t &\propto \int_{I_t} \left(\left.\frac{\partial U}{\partial x}\right|_{x=x_0+\Delta x} - \left.\frac{\partial U}{\partial x}\right|_{x=x_0-\Delta x} \right) \mathrm{d}t \\ &= \int_{I_t}\int_{I_x} \frac{\partial^2 U}{\partial x^2} \mathrm{d}x\mathrm{d}t \end{aligned}$$

from which it follows that the difference two functions being integrated is in fact identically 0. Then noting that energy and temperature are proportional, the result follows.

Example 5.6. Consider an infinitely long bar heated at one end; then we can set

$$T(x,0) = 0$$

so that the temperature is initially 0 everywhere, and heat the end to keep it at a constant temperature:

$$T(0,t) = H(t) = \begin{cases} 0 & t < 0 \\ Q & t > 0 \end{cases}$$

We also assume that at any fixed time t, $T(x,t) \to 0$ as $x \to \infty$.

In the last section, we found that a clever change of variables transformed the PDE into another PDE which was easier to solve. This problem does not admit such a natural change of variables, however. We also found that it was possible to transform a PDE to an ODE by finding a characteristic, or path along which y is invariant. Whilst we cannot do quite the same thing here, we can adopt another technique, and find a *similarity solution.* The basic idea is to find transformations of variables under which the equation and initial conditions are invariant, and to deduce the form of a solution in terms of one function, which can then be found by solving a single ODE. If you are not interested in the derivation, feel free to skip over the following argument.

Consider the *dilation* transformation

$$x' = \lambda^a x, \quad t' = \lambda^b t, \quad T' = \lambda^c T$$

The terms in the heat equation transform as follows:

$$\begin{aligned} \frac{\partial T}{\partial t} &= \lambda^{b-c}\frac{\partial T'}{\partial t'} \\ \frac{\partial^2 T}{\partial x^2} &= \lambda^{2a-c}\frac{\partial^2 T'}{\partial x'^2} \end{aligned}$$

So

$$\lambda^{b-c}\frac{\partial T'}{\partial t'} = \alpha\lambda^{2a-c}\frac{\partial^2 T'}{\partial x'^2}$$

which means that the equation is invariant if $b - c = 2a - c$ (that is, $b = 2a$); by this we mean that $T(x, t)$ solves the equation if and only if $T'(x', t')$ does too.

Now note that we can also find combinations of T, x and t which are invariant (for example by considering $x^i t^j T^k = (x')^i (t')^j (T')^k = \lambda^{ai+bj+ck} x^i t^j T^k$ and solving $ai + bj + ck = 0$). In particular

$$\begin{aligned} (T')(t')^{-c/b} &= \lambda^c T\left[\lambda^b t\right]^{-c/b} = Tt^{-c/b} \\ x'(t')^{-a/b} &= \lambda^a x\left[\lambda^b\right]^{-a/b} = xt^{-a/b} \end{aligned}$$

are invariant for any a, b, c. Since we have established $b = 2a$, it follows that

$$\xi = \frac{x'}{\sqrt{t'}} = \frac{x}{\sqrt{t}}$$

is invariant. This suggests looking for a solution of the form

$$\begin{aligned} \frac{T'}{(t')^{c/b}} &= \frac{T}{t^{c/b}} = \theta(\xi) \\ T &= t^{c/2a}\theta(\xi) \end{aligned}$$

In this case, we find

$$\begin{aligned} \frac{\partial T}{\partial t} &= \frac{c}{2a}t^{(c/2a)-1}\theta(\xi) + t^{c/2a}\theta'(\xi)\frac{\partial \xi}{\partial t} \\ &= t^{(c/2a)-1}\left[\frac{c}{2a}\theta(\xi) + t\theta'(\xi)\left(-\frac{1}{2}\frac{x}{\sqrt{t^3}}\right)\right] \\ &= t^{(c/2a)-1}\left[\frac{c}{2a}\theta(\xi) - \frac{\xi}{2}\theta'(\xi)\right] \end{aligned}$$

and

$$
\begin{aligned}
\frac{\partial T}{\partial x} &= t^{c/2a}\theta'(\xi)\frac{\partial \xi}{\partial x} \\
&= t^{(c/2a)-(1/2)}\theta'(\xi) \\
\frac{\partial^2 T}{\partial x^2} &= t^{(c/2a)-(1/2)}\theta''(\xi)\frac{\partial \xi}{\partial x} \\
&= t^{(c/2a)-1}\theta''(\xi)
\end{aligned}
$$

Hence the PDE has become

$$
\begin{aligned}
t^{(c/2a)-1}\left[\frac{c}{2a}\theta(\xi) - \frac{\xi}{2}\theta'(\xi)\right] &= \alpha t^{(c/2a)-1}\theta''(\xi) \\
\alpha\theta''(\xi) + \frac{\xi}{2}\theta'(\xi) - \frac{c}{2a}\theta(\xi) &= 0
\end{aligned}
$$

which is an ODE for $\theta(\xi)$.

So all we need to do is make a choice for a and c (in fact, this amounts to only one choice, because only their ratios are significant). We do this by seeing what combination of them give invariant boundary conditions, which depends on the nature of the boundary conditions.

In this case, the boundary condition is $T(0,t) = Q$. Note that $T' = t^{c/2a}\theta(0)$ which is only constant if and only if $c = 0$. Then the above equation becomes

$$
\alpha\theta''(\xi) + \frac{\xi}{2}\theta'(\xi) = 0
$$

where

$$
T = t^{c/2a}\theta(\xi) = \theta\left(\frac{x}{\sqrt{t}}\right)
$$

To summarize, we are attempting to find solutions

$$
\boxed{T(x,t) = \theta\left(\frac{x}{\sqrt{t}}\right)}
$$

which - as you can verify directly - is a solution of the equation iff θ satisfies

$$
\alpha\theta''(\xi) + \frac{\xi}{2}\theta'(\xi) = 0
$$

Now this equation can be solved using the integrating factor $\mu = \exp\left[\int \frac{\xi}{2\alpha}\mathrm{d}\xi\right] = e^{\xi^2/4\alpha}$, since then

$$
\begin{aligned}
\left(e^{\xi^2/4\alpha}\theta'\right)' &= 0 \\
\theta' &= Ae^{-\xi^2/4\alpha} \\
\theta &= A\int e^{-\xi^2/4\alpha}\mathrm{d}\xi + B \\
&= C\mathrm{erf}\left(\frac{\xi}{2\sqrt{\alpha}}\right) + B
\end{aligned}
$$

where we use:

Definition 5.7. The *error function* erf is defined by

$$\operatorname{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^z e^{-t^2} \mathrm{d}t$$

and has no closed form. (See Figure 5.4.)

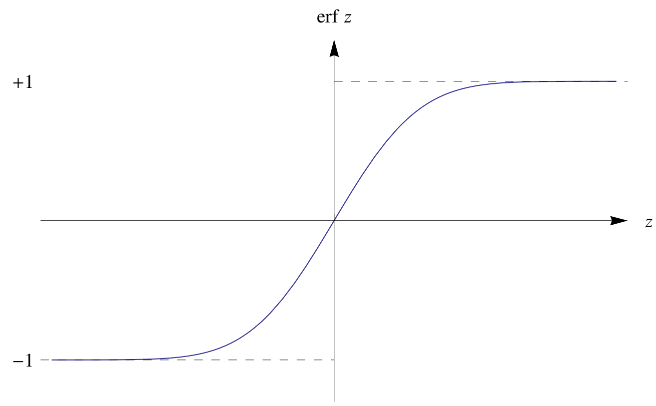


Figure 5.4: The error function $\operatorname{erf}(z)$

Hence for the point-like boundary condition, we have (for $t > 0$) that

$$T(0,t) = \theta(0) = Q$$

which implies that $B = Q$.

Also, $T(x,t) \to 0$ as $t \to 0$ (from above) so $\theta(\xi) \to 0$ as $\xi = x/\sqrt{t} \to \infty$. Therefore, because $\operatorname{erf}(z) \to 1$ as $z \to \infty$, it follows that $C = -1$, and so

$$\theta(\xi) = Q\left[1 - \operatorname{erf}\left(\frac{\xi}{2\sqrt{\alpha}}\right)\right]$$

It follows that

$$\begin{aligned} T(x,t) &= Q\left[1 - \operatorname{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)\right] \\ &= Q\operatorname{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) \end{aligned}$$

where:

Definition 5.8. The *complementary error function* erfc is defined by

$$\operatorname{erfc}(z) \equiv 1 - \operatorname{erf}(z)$$

Hence in this problem, the temperature curve at any fixed time t is given by a curve like $Q\operatorname{erfc}(x)$, except stretched horizontally by a factor $\sqrt{\alpha t}$. Over time, the shape is self-similar (hence *similarity solution*), being stretched like $\sqrt{t}$ as time passes. A few curves are shown in Figure 5.5.

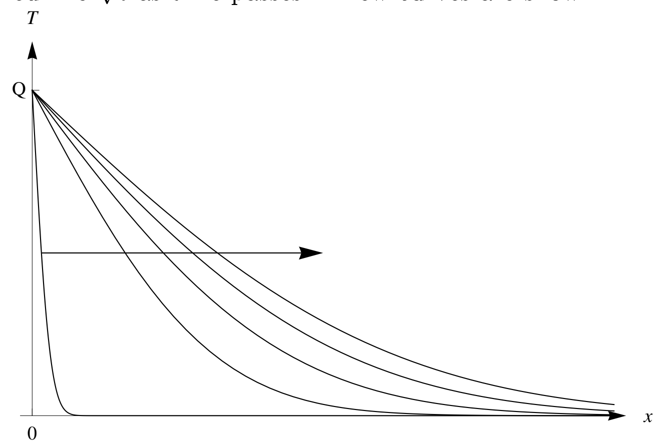


Figure 5.5: The temperature distribution with 5 equally spaced samples

6 * Generalized Methods for Ordinary Differential Equations

SECTION INCOMPLETE.

6.1 Systems of Linear Equations

6.2 First-Order Vector Equations

In section 4.7, we looked at various equations of the form

$$\dot{\mathbf{Y}} = M(t)\,\mathbf{Y} + \mathbf{F}(t)$$

for an unknown vector $\mathbf{Y}(t)$. How do we solve this for a general vector $\mathbf{Y}(t) \in \mathbb{R}^n$, general matrix $M(t)$ and general forcing term $\mathbf{F}(t)$?

The first thing we will do is look at the simplest case in $\mathbb{R}^n$:

$$\dot{\mathbf{Y}} = M\mathbf{Y}, \qquad \text{constant } M$$

For the one-dimensional case $\dot{y} = my$, we know the solution is $y = A\exp(mt)$. Why does this work? The key idea is that $\mathrm{d}(\exp t)/\mathrm{d}t = \exp t$, a special property of the function $\exp t$.

6.2.1 Matrix exponentials

But now we have to somehow include matrices into the answer. We could approach this is various ways; one is to think about diagonalizing the M matrix, if possible; then we find that some transformed version of $\mathbf{Y}$ will obey some simplified pair of separated equations. However, this leads us away into worrying about diagonalization, which is an unnecessary complication.

Instead, let's think about how we might try to define an exponential of a matrix, so that the chain rule gives "$\mathrm{d}(\exp Mt)/\mathrm{d}t = M\exp Mt$". So how can we define exp? Commonly, we define these functions in terms of their *Taylor series* - recall

$$\exp t = 1 + t + \frac{1}{2}t^2 + \frac{1}{3!}t^3 + \cdots = \sum_{m=0}^{\infty}\frac{t^m}{m!}$$

Can we simply shove a matrix into this calculation? We can certainly let T be a $n \times n$ matrix and write down

$$\exp T = I + T + \frac{1}{2}T^2 + \frac{1}{3!}T^3 + \cdots = \sum_{m=0}^{\infty}\frac{T^m}{m!}$$

since we know how to calculate powers of a matrix by multiplying it together - the only question we'd worry about is whether the sum *converges* to a well-defined matrix. (What do we mean by convergence? Simply that every component of the matrix we're making up converges individually. Any other resonable notions of convergence are equivalent.) This is not particularly hard to show, but not very relevant - the key idea is you still can't grow very rapidly term by term.

Proof. Let A be the magnitude of the largest element in the matrix T. Then note that if B similarly bounds some other matrix T', we have

$$\left|(TT')_{ik}\right| = \left|\sum_{j=1}^{n}\left(T_{ij}T'_{jk}\right)\right| \leq \sum_{j=1}^{n}\left|T_{ij}T'_{jk}\right| \leq n \cdot AB$$

Therefore, T^2 has elements no larger in magnitude than nA^2; T^3 is bounded by n^2A^3; and by induction T^m is bounded by n^mA^m (adding a factor of n for simplicity). As a result, each element of our sum defining $\exp T$ is bounded by $(nA)^m/m!$ - this means the series for each element is ***absolutely convergent***, since we know $\exp(nA)$ is well-defined. It is a theorem - from the course *Analysis I* - that this implies the series is converegnt. □

Anyhow, we can take the definition

$$\exp T = I + T + \frac{1}{2}T^2 + \frac{1}{3!}T^3 + \cdots = \sum_{m=0}^{\infty}\frac{T^m}{m!}$$

and run with it. Let $T = Mt$. Then (technically using more *Analysis I* theorems on differentiating infinite series, but let us live dangerously)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\exp(Mt) &= \frac{\mathrm{d}}{\mathrm{d}t}\sum_{m=0}^{\infty} M^m\frac{t^m}{m!} \\
&= \sum_{m=0}^{\infty} M^m\frac{\mathrm{d}}{\mathrm{d}t}\frac{t^m}{m!} \\
&= \sum_{m=1}^{\infty} M^m\frac{t^{m-1}}{(m-1)!} \\
&= \sum_{m=0}^{\infty} M^{m+1}\frac{t^m}{m!} \\
&= M\exp(Mt)
\end{aligned}$$

But this means that we can solve

$$\dot{\mathbf{Y}} = M\mathbf{Y}$$

straightforwardly by

$$\boxed{\mathbf{Y} = \exp(Mt)\,\mathbf{Y}_0}$$

noting that

$$\exp\begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} = I$$

6.2.2 The inhomogeneous case

Suppose

$$\dot{\mathbf{Y}} = M\mathbf{Y} + \mathbf{F}(t)$$

where M is still a constant matrix. Checking first the one-dimensional case, we find $\dot{y} = my + f$ can be solved by using an integrating factor:

$$\begin{aligned}
(\dot{y} - my - f)\, e^{-mt} &= 0 \\
\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-mt} y\right) - f e^{-mt} &= 0 \\
y &= e^{mt}\left[\text{const.} + \int^{t} e^{-mt'} f(t')\ \mathrm{d}t'\right]
\end{aligned}$$

Can we use a similar trick with matrices? Noting $M \exp(Mt) = \exp(Mt)\, M$ - which is obvious from the series definition - we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\exp(-Mt)\,\mathbf{Y}\right) &= -M\exp(-Mt)\,\mathbf{Y} + \exp(-Mt)\,\dot{\mathbf{Y}} \\
&= \exp(-Mt)\left[\dot{\mathbf{Y}} - M\mathbf{Y}\right] \\
&= \exp(-Mt)\,\mathbf{F}(t)
\end{aligned}$$

which we can solve by noting that

$$[\exp(Mt)]^{-1} = \exp(-Mt)$$

(which can be shown by computing $\exp(Mt)\exp(-Mt)$ in the series expansion) so that

$$\mathbf{Y} = \exp(Mt)\left[\text{const.} + \int^{t} \exp(-Mt')\,\mathbf{F}(t')\,\mathrm{d}t'\right]$$

6.2.3 The non-autonomous case?

So what if $M = M(t)$ is time-dependent? Let's return to the one-dimensional case to begin with, $\dot{y} = m(t)\, y$. This equation is separable:

$$\begin{aligned}
\frac{\dot{y}}{y} = m(t) \qquad &\Longrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}t}\ln y = m(t) \\
&\Longrightarrow \qquad y = y_0 \exp\left(\int^{t} m(t)\ \mathrm{d}t'\right)
\end{aligned}$$

which clearly agrees with the $m =$ const. case. Rather than worrying about the intermediate steps, let us try generalizing the answer naïvely:

$$\mathbf{Y} = \left[\exp\int^{t} M(t)\ \mathrm{d}t'\right]\mathbf{Y}_0$$

(The integral of a matrix is formed by integrating it component by component.)

What happens? Let $N = \int^t M \mathrm{d}t'$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \exp N &= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{m=0}^{\infty} \frac{N^m}{m!} \\
&= \frac{\mathrm{d}}{\mathrm{d}t} \sum_{m=0}^{\infty} \frac{N \cdot N \cdot \dots \cdot N}{m!} \\
&= \sum_{m=0}^{\infty} \frac{M \cdot N \cdot \dots \cdot N + N \cdot M \cdot \dots \cdot N + \dots + N \cdot N \cdot \dots \cdot M}{m!}
\end{aligned}$$

The problem is that if $NM \neq MN$, then the numerator is not necessarily the same as $m \cdot MN^{m-1}$.

Example 6.1. Consider

$$M = \begin{pmatrix} 1 & 0 \\ 2t & 0 \end{pmatrix}$$

Then

$$N = \begin{pmatrix} t & 0 \\ t^2 & 0 \end{pmatrix}$$

and so

$$\exp N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} t & 0 \\ t^2 & 0 \end{pmatrix} + \frac{1}{2!} \begin{pmatrix} t^2 & 0 \\ t^3 & 0 \end{pmatrix} + \frac{1}{3!} \begin{pmatrix} t^3 & 0 \\ t^4 & 0 \end{pmatrix} + \dots = \begin{pmatrix} e^t & 0 \\ t\left(e^t - 1\right) & 1 \end{pmatrix}$$

but then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \exp N &= \boxed{\begin{pmatrix} e^t & 0 \\ e^t\left(1+t\right) - 1 & 0 \end{pmatrix}} \\
M \exp N &= \begin{pmatrix} 1 & 0 \\ 2t & 0 \end{pmatrix} \begin{pmatrix} e^t & 0 \\ t\left(e^t - 1\right) & 1 \end{pmatrix} \\
&= \boxed{\begin{pmatrix} e^t & 0 \\ 2te^t & 0 \end{pmatrix}}
\end{aligned}$$

You can easily check that this means you do not get the correct solution to a typical initial value problem.

In fact, sadly we *cannot* solve this problem simply in general.

6.3 Degeneracy

We noted on a few occasions that there is a general principle at work for linear equations with constant coefficients: if $y_c(x) = e^{\lambda x}$ is some single solution to the homogeneous equation, then $x \cdot y_c$ has a remainder under the differential operator which is a multiple of $y_c = e^{\lambda x}$. This allowed us to solve second-order degenerate equations.

In fact, we can prove something slightly more general and very useful:

Lemma 6.2. *For an nth-order forced linear equation, if $y_c(x)$ is some single solution to the complementary equation, then applying the differential operator to $x \cdot y_c$ gives*

$$D[xy_c] = E[y_c]$$

where E is another differential operator with related coefficients

$$E[y] = \sum_i ic_i(y)^{(i-1)}$$

Proof. We have by definition that

$$D[xy_c] = \sum_i c_i (xy_c)^{(i)}$$

Now by Leibniz's product rule,

$$\begin{aligned}(xy_c)^{(i)} &= x \cdot (y_c)^{(i)} + i \cdot 1 \cdot (y_c)^{(i-1)} + 0 \cdot (\cdots) \\ &= x \cdot (y_c)^{(i)} + i \cdot (y_c)^{(i-1)}\end{aligned}$$

so

$$\begin{aligned}D[xy_c] &= \sum_i c_i \left[x \cdot (y_c)^{(i)} + i \cdot (y_c)^{(i-1)}\right] \\ &= x \cdot \underbrace{D[y_c]}_{0} + \sum_i ic_i(y_c)^{(i-1)} \\ &= \sum_i ic_i(y_c)^{(i-1)} \\ &= E[y_c]\end{aligned}$$

where we used the fact that y_c solves the equation $D[y_c] = 0$ to simplify the expression. Note that E is another differential operator, of the reduced order $n-1$, dependent only on the coefficients of the original equation (that is, independent of y_c). □

We can deduce several results from this.

Theorem 6.3. *If λ is a repeated root of multiplicity m of the characteristic equation of the operator $D[\cdot]$, which is linear with constant coefficients, then*

$$y = e^{\lambda x}, xe^{\lambda x}, x^2 e^{\lambda x}, \cdots, x^{m-1}e^{\lambda x}$$

are all linearly independent solutions to the equation $D[y] = 0$.

Proof. The characteristic equation of D is

$$g(\mu) = \sum c_i \mu^i = 0$$

We proceed by induction - assume that the result holds for multiplicity $m-1$, and assume λ is a solution of multiplicity $m > 1$. Then the first derivative of g at λ is zero - in fact, it is a root of multiplicity $m-1$. To see this, note that by definition

$$g(\mu) = (\mu - \lambda)^m h(\mu)$$

for some polynomial $h(\mu)$, and then

$$g'(\mu) = (\mu - \lambda)^{m-1} [m h(\mu) + h'(\mu)]$$

But we know that

$$g'(\mu) = \sum i c_i \mu^i$$

which exactly the characteristic equation of the operator $E[\cdot]$. In fact, as $y_1 = x^{m-2} e^{\lambda x}$ satisfies $D[y_1] = 0$ by the induction hypothesis, we have

$$\begin{aligned} D\left[x^{m-1} e^{\lambda x}\right] &= D[x \cdot y_1] \\ &= E[y_1] \\ &= E\left[x^{m-2} e^{\lambda x}\right] \end{aligned}$$

Then once more, the induction hypothesis tells us that because λ is a root of E's characteristic equation with multiplicity $m-1$, $y_1 = x^{m-2} e^{\lambda x}$ is a solution to $E[y] = 0$. Hence

$$D\left[x^{m-1} e^{\lambda x}\right] = 0$$

Now we are done by induction. □

So we always have families of solutions of the form

$$y = e^{\lambda x}, x e^{\lambda x}, x^2 e^{\lambda x}, \cdots, x^{m-1} e^{\lambda x}$$

for linear equations with constant coefficients.

6.4 The Wronskian and Abel's Theorem

Take any homogeneous, linear nth-order differential equation, which can be written in the form

$$\frac{\mathrm{d}^n y}{\mathrm{d}x^n} + p_{n-1}(x)\frac{\mathrm{d}^{n-1} y}{\mathrm{d}x^{n-1}} + \cdots + p_1(x)\frac{\mathrm{d}y}{\mathrm{d}x} + p_0(x)\, y = 0$$

for some set of x on the real line $\mathbb{R}$.

Theorem 6.4 (Abel's Identity). *Let us $p_{n-1}(x)$ be a continuous function. Then on this set, the Wronskian $W(x)$ of a set of n solutions to this equation obeys the equation*

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x} p_{n-1}(t)\,\mathrm{d}t\right)$$

We shall prove this general statement, which has the corollary that $W(x_0) \neq 0$ implies $W(x) \neq 0$ for all x - that is, *Abel's Theorem.*

Proof. The derivative of the Wronskian W is the derivative of a matrix determinant. Remembering that (from the Leibniz formula) we can express the determinant as a sum over permutations of the columns like so:

$$\det A = \sum_{\sigma} \operatorname{sgn}(\sigma)\, A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{m\sigma(m)}$$

we can see that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\det A &= \sum_{\sigma} \operatorname{sgn}(\sigma)\, A'_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{m\sigma(m)} \\
&\quad + \sum_{\sigma} \operatorname{sgn}(\sigma)\, A_{1\sigma(1)} A_{2\sigma(2)'} \cdots A_{m\sigma(m)} \\
&\quad + \cdots \\
&\quad + \sum_{\sigma} \operatorname{sgn}(\sigma)\, A_{1\sigma(1)} A_{2\sigma(2)} \cdots A'_{m\sigma(m)}
\end{aligned}$$

and so if the solutions are $y_1, \cdots, y_n$ then

$$W' = \begin{vmatrix}
y_1' & y_2' & \cdots & y_n' \\
y_1' & y_2' & \cdots & y_n' \\
y_1'' & y_2'' & \cdots & y_n'' \\
y_1''' & y_2''' & \cdots & y_n''' \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix} + \begin{vmatrix}
y_1 & y_2 & \cdots & y_n \\
y_1'' & y_2'' & \cdots & y_n'' \\
y_1'' & y_2'' & \cdots & y_n'' \\
y_1''' & y_2''' & \cdots & y_n''' \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)}
\end{vmatrix}$$

$$+ \cdots + \begin{vmatrix}
y_1' & y_2' & \cdots & y_n' \\
y_1'' & y_2'' & \cdots & y_n'' \\
\vdots & \vdots & \ddots & \vdots \\
y_1^{(n-3)} & y_2^{(n-3)} & \cdots & y_n^{(n-3)} \\
y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\
y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)}
\end{vmatrix}$$

But then every matrix except for the last has two identical rows, and therefore has determinant 0. So

$$W' = \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n)} & y_2^{(n)} & \cdots & y_n^{(n)} \end{vmatrix}$$

But now, using our expression for $y_i^{(n)}$ in terms of $y_i^{(n-k)}$ we can subtract p_0 times the first row, p_1 times the second, and so on, from the last row. This gives us

$$\begin{aligned} W' &= \begin{vmatrix} y_1' & y_2' & \cdots & y_n' \\ y_1'' & y_2'' & \cdots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ -p_{n-1}y_1^{(n-1)} & -p_{n-1}y_2^{(n-1)} & \cdots & -p_{n-1}y_n^{(n-1)} \end{vmatrix} \\ &= -p_{n-1}W \end{aligned}$$

Then the result follows immediately, because we can integrate W'/W to get $\ln W$ and then

$$W = e^{-\int_{x_0}^{x} p_{n-1}(t)\mathrm{d}t} W_0$$

taking account of the various factors, and noting that the integral should be 0 at $x = x_0$. □

6.5 Variation of Parameters